Spinor Tech Summary

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Abstract

We summarize some definitions and identities on the spinorial technique in writing the scattering amplitudes.

1 Two-spinors and Basic definitions

Here

$$k^2 \equiv k^\mu k_\mu \equiv \eta_{\mu\nu} k^\mu k^\nu \equiv (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 = 0 \tag{1}$$

$$k_0 = k^0\,,\ k_i = -k^i \quad (i = 1, 2, 3) \tag{2}$$

$$|j^\pm\rangle \equiv u_\pm(k_j)\,, \qquad \langle j^\pm| \equiv \overline{u_\pm}(k_j) \tag{3}$$

$$\langle ij\rangle \equiv \langle i^-|j^+\rangle = \overline{u_-}(k_i) u_+(k_j) \tag{4}$$

$$[ij] \equiv \langle i^+|j^-\rangle = \overline{u_+}(k_i) u_-(k_j) \tag{5}$$

$$u_+(k) = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}e^{i\phi_k} \end{pmatrix}, \qquad u_-(k) = \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix} \tag{6}$$

where

$$e^{\pm i\phi_k} = \frac{k^1 \pm ik^2}{\sqrt{(k^1)^2 + (k^2)^2}} = \frac{k^1 \pm ik^2}{\sqrt{k_+ k_-}}\,, \qquad k_\pm = k^0 \pm k^3 \tag{7}$$

$$\overline{u_+}(k) = u_+^\dagger(k) = \left(\sqrt{k_+}\,,\ \sqrt{k_-}e^{-i\phi_k}\right)\,, \quad \overline{u_-}(k) = u_-^\dagger(k) = \left(\sqrt{k_-}e^{i\phi_k}\,,\ -\sqrt{k_+}\right) \tag{8}$$

$$\langle ij\rangle = \sqrt{k_{i-}k_{j+}}e^{i\phi_{k_i}} - \sqrt{k_{i+}k_{j-}}e^{i\phi_{k_j}} \tag{9}$$

$$[ij] = \sqrt{k_{i+}k_{j-}}e^{-i\phi_{k_j}} - \sqrt{k_{i-}k_{j+}}e^{-i\phi_{k_i}} \tag{10}$$

$$\langle ij\rangle[ji] = 2k_i \cdot k_j \tag{11}$$

$$|\langle ij\rangle| = |[ij]| = \sqrt{2k_i \cdot k_j} \tag{12}$$

$$\langle ji\rangle = -\langle ij\rangle\,, \qquad [ji] = -[ij] \tag{13}$$

especially,

$$\langle ii\rangle = \langle i^-|i^+\rangle = \overline{u_-}(k_i)u_+(k_i) = 0 \tag{14}$$
$$[ii] = \langle i^+|i^-\rangle = \overline{u_+}(k_i)u_-(k_i) = 0 \tag{15}$$

$$\begin{aligned} u_+(k)\overline{u_+}(k) &= \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}e^{i\phi_k} \end{pmatrix} \left(\sqrt{k_+}\,,\ \sqrt{k_-}e^{-i\phi_k}\right) \\ &= \begin{pmatrix} k_+ & \sqrt{k_+k_-}e^{-i\phi_k} \\ \sqrt{k_+k_-}e^{i\phi_k} & k_- \end{pmatrix} \\ &= \begin{pmatrix} k^0+k^3 & k^1-ik^2 \\ k^1+ik^2 & k^0-k^3 \end{pmatrix} \\ &= k^0\sigma^0 + k^i\sigma^i = k_0\sigma^0 - k_i\sigma^i \end{aligned} \tag{16}$$

$$\begin{aligned} u_-(k)\overline{u_-}(k) &= \begin{pmatrix} \sqrt{k_-}e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix} \left(\sqrt{k_-}e^{i\phi_k}\,,\ -\sqrt{k_+}\right) \\ &= \begin{pmatrix} k_- & -\sqrt{k_+k_-}e^{-i\phi_k} \\ -\sqrt{k_+k_-}e^{i\phi_k} & k_+ \end{pmatrix} \\ &= \begin{pmatrix} k^0-k^3 & -k^1+ik^2 \\ -k^1-ik^2 & k^0+k^3 \end{pmatrix} \\ &= k^0\sigma^0 - k^i\sigma^i = k_0\sigma^0 + k_i\sigma^i \end{aligned} \tag{17}$$

where

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

$$(k_0\sigma^0 \pm k_i\sigma^i)u_\pm(k) = 0 \qquad (\text{`}Dirac\ equation') \tag{19}$$

$$u_-(k) = i\sigma^2 u_+^*(k)\,, \qquad \overline{u_-}(k) = u_+^T(k)(-i)\sigma^2 \tag{20}$$

$$\overline{u_-}(k_i)u_-(k_j) = u_+^T(k_i)(-i)\sigma^2 i\sigma^2 u_+^*(k_j) = u_+^T(k_i)u_+^*(k_j) = \overline{u_+}(k_j)u_+(k_i) \tag{21}$$

$$\overline{u_-}(k_i)\sigma^i u_-(k_j) = u_+^T(k_i)(-i)\sigma^2\sigma^i i\sigma^2 u_+^*(k_j) = -u_+^T(k_i)(\sigma^i)^T u_+^*(k_j) = -\overline{u_+}(k_j)\sigma^i u_+(k_i) \tag{22}$$

Fierz identity

$$\sigma^\mu = (1, \sigma^i) \tag{23}$$

$$(\sigma^\mu)_{ab}(\sigma_\mu)_{cd} = \delta_{ab}\delta_{cd} - (\sigma^i)_{ab}(\sigma^i)_{cd} = -2(\sigma^2)_{ac}(\sigma^2)_{bd} \tag{24}$$

$$\begin{aligned}\langle i^+|\sigma^\mu|j^+\rangle\langle p^+|\sigma_\mu|q^+\rangle &= u_+^\dagger(k_i)\sigma^\mu u_+(k_j)u_+^\dagger(k_p)\sigma_\mu u_+(k_q) \\ &= -2u_+^\dagger(k_i)u_-(k_p)u_-^\dagger(k_j)u_+(k_q) \\ &= -2\langle jq\rangle[ip] \end{aligned} \tag{25}$$

<u>Schouten identity</u>

$$\epsilon_{ab}\epsilon_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \tag{26}$$

$$\epsilon_{ac}\epsilon_{db} = \delta_{ad}\delta_{cb} - \delta_{ab}\delta_{cd} \tag{27}$$

$$\epsilon_{ad}\epsilon_{bc} = \delta_{ab}\delta_{dc} - \delta_{ac}\delta_{db} \tag{28}$$

Thus

$$\epsilon_{ab}\epsilon_{cd} + \epsilon_{ac}\epsilon_{db} + \epsilon_{ad}\epsilon_{bc} = 0 \tag{29}$$

Then

$$\langle ij\rangle\langle k\ell\rangle + \langle i\ell\rangle\langle jk\rangle + \langle ik\rangle\langle \ell j\rangle = 0 \tag{30}$$

<u>"eikonal" identity</u>

$$\sum_{i=j}^{k-1} \frac{\langle i\,(i+1)\rangle}{\langle iq\rangle\langle q\,(i+1)\rangle} = \frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} \tag{31}$$

which can be proved by

$$\begin{aligned}\frac{\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} + \frac{\langle k\,(k+1)\rangle}{\langle kq\rangle\langle q\,(k+1)\rangle} &= \frac{\langle jk\rangle\langle q\,(k+1)\rangle - \langle jq\rangle\langle k\,(k+1)\rangle}{\langle jq\rangle\langle qk\rangle\langle q\,(k+1)\rangle} \\ &= \frac{-\langle kj\rangle\langle q\,(k+1)\rangle - \langle jq\rangle\langle k\,(k+1)\rangle}{\langle jq\rangle\langle qk\rangle\langle q\,(k+1)\rangle} \\ &= \frac{\langle qk\rangle\langle j\,(k+1)\rangle}{\langle jq\rangle\langle qk\rangle\langle q\,(k+1)\rangle} \\ &= \frac{\langle j\,(k+1)\rangle}{\langle jq\rangle\langle q\,(k+1)\rangle} \end{aligned} \tag{32}$$

where Schouten identity is used.

2 Four-component spinors

gamma matrices

Bjorken-Drell rep.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{33}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,, \quad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0\,, \quad (\gamma^5)^2 = 1 \tag{34}$$

four component spinor

$$\mathcal{U}_+(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} u_+(k) \\ u_+(k) \end{pmatrix}, \qquad \mathcal{U}_-(k) = \frac{1}{\sqrt{2}}\begin{pmatrix} u_-(k) \\ -u_-(k) \end{pmatrix} \tag{35}$$

$$\gamma^5\mathcal{U}_\pm(k) = \pm\mathcal{U}_\pm(k) \tag{36}$$

$$\overline{\mathcal{U}_+}(k) = \mathcal{U}_+^\dagger(k)\gamma^0 = \frac{1}{\sqrt{2}}\left(\overline{u_+}(k)\,, -\overline{u_+}(k)\right) \tag{37}$$

$$\overline{\mathcal{U}_-}(k) = \mathcal{U}_-^\dagger(k)\gamma^0 = \frac{1}{\sqrt{2}}\left(\overline{u_-}(k)\,, \overline{u_-}(k)\right) \tag{38}$$

$$\overline{\mathcal{U}_\pm}(k)\gamma^5 = \mp\overline{\mathcal{U}_\pm}(k) \tag{39}$$

$$\overline{\mathcal{U}_\pm}(k_i)\gamma^0\mathcal{U}_\pm(k_j) = \overline{u_\pm}(k_i)u_\pm(k_j) \tag{40}$$

$$\overline{\mathcal{U}_\pm}(k_i)\gamma^i\mathcal{U}_\pm(k_j) = \pm\overline{u_\pm}(k_i)\sigma^i u_\pm(k_j) \tag{41}$$

$$\begin{aligned}\overline{\mathcal{U}_\mp}(k_i)\gamma^\mu\mathcal{U}_\pm(k_j) &= \pm\overline{\mathcal{U}_\mp}(k_i)\gamma^\mu\gamma^5\mathcal{U}_\pm(k_j) = \mp\overline{\mathcal{U}_\mp}(k_i)\gamma^5\gamma^\mu\mathcal{U}_\pm(k_j) \\ &= -\overline{\mathcal{U}_\mp}(k_i)\gamma^\mu\mathcal{U}_\pm(k_j) = 0\end{aligned} \tag{42}$$

$$|j^\pm\rangle \equiv \mathcal{U}_\pm(k_j)\,, \quad \langle j^\pm| \equiv \overline{\mathcal{U}_\pm}(k_j) \quad \textit{Note! the same notation as 2spinors} \tag{43}$$

$$\langle i^\pm|j^\pm\rangle = \overline{\mathcal{U}_\pm}(k_i)\mathcal{U}_\pm(k_j) = 0 \tag{44}$$

$$\langle i^-|j^+\rangle = \overline{\mathcal{U}_-}(k_i)\mathcal{U}_+(k_j) = \langle ij\rangle\,, \qquad \langle i^+|j^-\rangle = \overline{\mathcal{U}_+}(k_i)\mathcal{U}_-(k_j) = [ij] \tag{45}$$

$$\langle i^\pm|\gamma^\mu|i^\pm\rangle = \overline{\mathcal{U}_\pm}(k_i)\gamma^\mu\mathcal{U}_\pm(k_i) = 2k_i^\mu \tag{46}$$

$$\langle i^-|\gamma^\mu|j^-\rangle = \overline{\mathcal{U}_-}(k_i)\gamma^\mu\mathcal{U}_-(k_j) = \overline{\mathcal{U}_+}(k_j)\gamma^\mu\mathcal{U}_+(k_i) = \langle j^+|\gamma^\mu|i^+\rangle \tag{47}$$

$$\langle i^\mp|\gamma^\mu|j^\pm\rangle = \overline{\mathcal{U}_\mp}(k_i)\gamma^\mu\mathcal{U}_\pm(k_j) = 0 \tag{48}$$

Fierz identity

$$\langle i^-|\gamma^\mu|j^-\rangle\langle p^+|\gamma_\mu|q^+\rangle = \langle j^+|\gamma^\mu|i^+\rangle\langle p^+|\gamma_\mu|q^+\rangle = -2\langle iq\rangle[jp] \tag{49}$$

$$
\begin{aligned}
\langle i^-|\gamma^\mu|j^-\rangle\gamma_\mu &= 2(|j^-\rangle\langle i^-| + |i^+\rangle\langle j^+|) && (50)\\
\langle i^+|\gamma^\mu|j^+\rangle\gamma_\mu &= 2(|j^+\rangle\langle i^+| + |i^-\rangle\langle j^-|) && (51)
\end{aligned}
$$

'Dirac equation'

$$
\not{k}\,\mathcal{U}_\pm(k) = \not{k}|k^\pm\rangle = 0 \qquad (52)
$$

Projector

$$
\begin{aligned}
\mathcal{U}_\pm(k)\overline{\mathcal{U}_\pm}(k) &= \frac{1}{2}\begin{pmatrix} u_\pm(k)\overline{u_\pm}(k) & \mp u_\pm(k)\overline{u_\pm}(k) \\ \pm u_\pm(k)\overline{u_\pm}(k) & -u_\pm(k)\overline{u_\pm}(k) \end{pmatrix} \\
&= \frac{1}{2}(1 \pm \gamma^5)\not{k} && (53)
\end{aligned}
$$

re-discover:

$$
\begin{aligned}
\langle ij\rangle[ji] &= \langle i^-|j^+\rangle\langle j^+|i^-\rangle = \mathrm{tr}\,\overline{\mathcal{U}_-}(k_i)\mathcal{U}_+(k_j)\overline{\mathcal{U}_+}(k_j)\mathcal{U}_-(k_i) \\
&= \mathrm{tr}\left(\frac{1-\gamma^5}{2}\not{k}_i\not{k}_j\right) = 2k_i\cdot k_j && (54)
\end{aligned}
$$

3 QED (QCD) Examples (I)

First of all, remember that

$$s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j = \langle ij\rangle[ji] \tag{55}$$

3.1 $e^-e^+ \to \mu^-\mu^+$

(Of course, the amplitude for $e^-e^+ \to q\bar{q}$ can be obtained similarly)

$$\mathcal{A}_4 = ie^2 A_4\ \delta(\sum_i k_i) \tag{56}$$

3.1.1 $e_L^-(1)e_R^+(2) \to \mu_R^-(3)\mu_L^+(4)$

$$A_4 = \frac{\langle 2^-|\gamma^\mu|1^-\rangle\langle 3^+|\gamma_\mu|4^+\rangle}{s_{12}} = -2\frac{\langle 24\rangle[13]}{s_{12}} \tag{57}$$

here we used (49).

$$|A_4| = 2\frac{|s_{13}|}{s_{12}} \propto 1 - \cos\theta \quad (CMF) \tag{58}$$

helicity suppressed as $1 \parallel 3$ or $2 \parallel 4$

$$A_4 = -2\frac{\langle 24\rangle[13]}{s_{12}} = -2\frac{\langle 24\rangle[13]\langle 13\rangle}{\langle 12\rangle[21]\langle 13\rangle} = 2\frac{\langle 24\rangle[24]\langle 24\rangle}{\langle 12\rangle[24]\langle 43\rangle} = -2\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle} \tag{59}$$

here we used

$$\sum_{j=1}^{4}\langle ij\rangle[jk] = 0 \ \ (derived\ from\ (16))\,, \quad s_{12} = s_{34}\,, \quad s_{13} = s_{24} \tag{60}$$

Exercise: Show

$$A_4 = -2\frac{[13]^2}{[12][34]} \tag{61}$$

3.1.2 $e_R^-(1)e_L^+(2) \to \mu_R^-(3)\mu_L^+(4)$

Using (47), we obtain

$$A_4 = +2\frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle} \tag{62}$$

3.1.3 $e_L^-(1)e_R^+(2) \to \mu_L^-(3)\mu_R^+(4)$

$$A_4 = +2\frac{\langle 23\rangle^2}{\langle 12\rangle\langle 34\rangle} \tag{63}$$

3.1.4 $e^-_R(1)e^+_L(2) \to \mu^-_L(3)\mu^+_R(4)$

$$A_4 = -2\frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle} \tag{64}$$

3.1.5 Unpolarized, helicity-summed cross sections

$$\begin{aligned}\frac{d\sigma(e^-e^+\to\mu^-\mu^+)}{d\cos\theta} &\propto \frac{1}{2}\sum_{helicity}|A_4|^2 = 4\left\{\left|\frac{\langle 24\rangle^2}{\langle 12\rangle\langle 34\rangle}\right|^2+\left|\frac{\langle 14\rangle^2}{\langle 12\rangle\langle 34\rangle}\right|^2\right\}\\ &= 4\frac{s_{13}^2+s_{14}^2}{s_{12}^2}\propto(1-\cos\theta)^2+(1+\cos\theta)^2\propto 1+\cos^2\theta\end{aligned}\tag{65}$$

$e^-e^+ \to \mu^-\mu^+$
$s_{12}=s,\ s_{13}=t,\ s_{14}=u$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{t^2+u^2}{s^2} \tag{66}$$

(In the limit of $t\to 0$, $\frac{d\sigma}{d\Omega}\to\frac{\alpha^2}{2s}$)

$e^-\mu^- \to e^-\mu^-$
$s_{12}=t,\ s_{13}=s,\ s_{14}=u$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\frac{s^2+u^2}{t^2} \tag{67}$$

(In the limit of $t\to 0$, $\frac{d\sigma}{d\Omega}\to\frac{\alpha^2 s}{t^2}$)

3.2 e^-e^-, e^-e^+ scattering

$e(1)e(2)\to e(3)e(4)$

$$A_{+--+} = \frac{\langle 3^-|\gamma^\mu|1^-\rangle\langle 4^+|\gamma_\mu|2^+\rangle}{s_{13}} = +2\frac{\langle 23\rangle[14]}{s_{13}} \tag{68}$$

$$A_{+-+-} = -\frac{\langle 4^-|\gamma^\mu|1^-\rangle\langle 3^+|\gamma_\mu|2^+\rangle}{s_{14}} = -2\frac{\langle 24\rangle[13]}{s_{14}} \quad (note\ the\ sign\ from\ exchange\ 3\leftrightarrow 4) \tag{69}$$

$$\begin{aligned}A_{++--} &= \frac{\langle 3^-|\gamma^\mu|1^-\rangle\langle 4^-|\gamma_\mu|2^-\rangle}{s_{13}} - \frac{\langle 4^-|\gamma^\mu|1^-\rangle\langle 3^-|\gamma_\mu|2^-\rangle}{s_{14}}\\ &= -2\langle 34\rangle[12]\left(\frac{1}{s_{13}}+\frac{1}{s_{14}}\right) = +2\langle 34\rangle[12]\left(\frac{s_{12}}{s_{13}s_{14}}\right)\end{aligned}\tag{70}$$

since $s_{12}+s_{13}+s_{14}=0$.

Unpolarized, helicity-summed cross sections

$$|A|^2 \propto \frac{s_{14}^2}{s_{13}^2} + \frac{s_{13}^2}{s_{14}^2} + \frac{s_{12}^4}{s_{13}^2 s_{14}^2} = \frac{s_{12}^4 + s_{13}^4 + s_{14}^4}{s_{13}^2 s_{14}^2} \tag{71}$$

For $e^-e^- \to e^-e^-$ or $e^+e^+ \to e^+e^+$ (Møller scattering),
$s_{12} = s$, $s_{13} = t$, $s_{14} = u$

$$|A|^2 \propto \frac{s^4 + t^4 + u^4}{t^2 u^2} \tag{72}$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{2s} \frac{s^4 + t^4 + u^4}{t^2 u^2} & (73) \\ &= \frac{\alpha^2}{2s} \left(\frac{s^2 + u^2}{t^2} + \frac{s^2 + t^2}{u^2} + 2\frac{s^2}{ut} \right) & (74) \\ &= \frac{\alpha^2}{4s} \left(\frac{s-u}{t} + \frac{s-t}{u} \right)^2 & (75) \end{aligned}$$

(In the limit of $t \to 0$, $\frac{d\sigma}{d\Omega} \to \frac{\alpha^2 s}{t^2}$)

For $e^-e^+ \to e^-e^+$ (Bhabha scattering),
$s_{13} = s$, $s_{14} = t$, $s_{12} = u$

$$|A|^2 \propto \frac{s^4 + t^4 + u^4}{s^2 t^2} \tag{76}$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\alpha^2}{2s} \frac{s^4 + t^4 + u^4}{s^2 t^2} & (77) \\ &= \frac{\alpha^2}{2s} \left(\frac{s^2 + u^2}{t^2} + \frac{u^2 + t^2}{s^2} + 2\frac{u^2}{st} \right) & (78) \\ &= \frac{\alpha^2}{4s} \left(\frac{s-u}{t} + \frac{t-u}{s} \right)^2 & (79) \end{aligned}$$

(In the limit of $t \to 0$, $\frac{d\sigma}{d\Omega} \to \frac{\alpha^2 s}{t^2}$)

4 Spinor-helicity rep. for polarization

4.1 Polarization vectors

$$\epsilon^+_\mu(k,q) = \frac{\langle q^-|\gamma_\mu|k^-\rangle}{\sqrt{2}\langle qk\rangle}\,, \qquad \epsilon^-_\mu(k,q) = \frac{\langle q^+|\gamma_\mu|k^+\rangle}{\sqrt{2}[kq]} \tag{80}$$

where $q\cdot k$ does not vanish.

$$k\cdot\epsilon^\pm(k,q) = 0 \qquad (Transverse)\ (from\ Dirac\ eq.) \tag{81}$$

$$\epsilon^+\cdot\epsilon^- = \frac{\langle q^-|\gamma^\mu|k^-\rangle\langle q^+|\gamma_\mu|k^+\rangle}{2\langle qk\rangle[kq]} = \frac{-2\langle qk\rangle[kq]}{2\langle qk\rangle[kq]} = -1 \tag{82}$$

$$\epsilon^+\cdot\epsilon^+ = \frac{\langle q^-|\gamma^\mu|k^-\rangle\langle q^-|\gamma_\mu|k^-\rangle}{2\langle qk\rangle^2} = \frac{\langle q^-|\gamma^\mu|k^-\rangle\langle k^+|\gamma_\mu|q^+\rangle}{2\langle qk\rangle^2} = 0 \tag{83}$$

$$\epsilon^+_\mu\epsilon^-_\nu + \epsilon^-_\mu\epsilon^+_\nu = -\eta_{\mu\nu} + \frac{k_\mu q_\nu + k_\nu q_\mu}{k\cdot q} \tag{84}$$

where we used (47), (53), (46) and $\gamma_\mu \not{q}\gamma_\nu + \gamma_\nu\not{q}\gamma_\mu = 2(q_\nu\gamma_\mu + q_\mu\gamma_\nu - \not{q}\eta_{\mu\nu})$. It is easy to confirm $k\cdot\epsilon^+\epsilon^-_\nu + k\cdot\epsilon^-\epsilon^+_\nu = q\cdot\epsilon^+\epsilon^-_\nu + q\cdot\epsilon^-\epsilon^+_\nu = 0$.

From (53),

$$\mathcal{U}_\pm(p)\overline{\mathcal{U}_\pm}(p) = \frac{1}{2}(1\pm\gamma^5)\not{p} \tag{85}$$

then for arbitrary p

$$p\cdot\epsilon^+(k,q) = \frac{\langle q^-|\not{p}|k^-\rangle}{\sqrt{2}\langle qk\rangle} = \frac{\langle qp\rangle[pk]}{\sqrt{2}\langle qk\rangle}\,, \quad p\cdot\epsilon^-(k,q) = \frac{\langle q^+|\not{p}|k^+\rangle}{\sqrt{2}[kq]} = \frac{[qp]\langle pk\rangle}{\sqrt{2}[kq]} \tag{86}$$

Thus we find

$$q\cdot\epsilon^\pm = 0 \quad (and\ rediscover \quad k\cdot\epsilon^\pm = 0) \tag{87}$$

Exercise. Show

$$\epsilon^+(k_1,q)\cdot\epsilon^+(k_2,q) = \epsilon^-(k_1,q)\cdot\epsilon^-(k_2,q) = 0 \tag{88}$$

$$\epsilon^+(k_1,q)\cdot\epsilon^-(k_2,k_1) = 0 \tag{89}$$

Actually, you must show

$$\epsilon^+(k,q)\cdot\epsilon^+(k',q') = -\frac{\langle qq'\rangle[kk']}{\langle qk\rangle\langle q'k'\rangle} \tag{90}$$

$$\epsilon^+(k,q)\cdot\epsilon^-(k',q') = -\frac{\langle qk'\rangle[kq']}{\langle qk\rangle[k'q']} \tag{91}$$

Using (51), we find

$$\not{\epsilon}^+(k,q) = \frac{\sqrt{2}}{\langle qk\rangle}(|k^-\rangle\langle q^-| + |q^+\rangle\langle k^+|)\,,\ \not{\epsilon}^-(k,q) = \frac{\sqrt{2}}{[kq]}(|k^+\rangle\langle q^+| + |q^-\rangle\langle k^-|) \tag{92}$$

Then

$$\not{\epsilon}^\pm(k,q)|q^\pm\rangle = 0\,, \qquad \langle q^\mp|\not{\epsilon}^\pm(k,q) = 0 \tag{93}$$

4.2 Significance of q

$$\begin{aligned}
\epsilon_\mu^+(k,q') &= \frac{\langle q'^-|\gamma_\mu|k^-\rangle}{\sqrt{2}\langle q'k\rangle} = \frac{\langle q'^-|\gamma_\mu|k^-\rangle\langle k^-|q^+\rangle}{\sqrt{2}\langle q'k\rangle\langle kq\rangle} = \frac{\langle q'^-|\gamma_\mu \not{k}|q^+\rangle}{\sqrt{2}\langle q'k\rangle\langle kq\rangle} \\
&= -\frac{\langle q'^-|\not{k}\gamma_\mu|q^+\rangle}{\sqrt{2}\langle q'k\rangle\langle kq\rangle} + \frac{\sqrt{2}\langle q'q\rangle}{\langle q'k\rangle\langle kq\rangle}k_\mu \\
&= \epsilon_\mu^+(k,q) + \frac{\sqrt{2}\langle q'q\rangle}{\langle q'k\rangle\langle kq\rangle}k_\mu \qquad (94)
\end{aligned}$$

It's a gauge degree of freedom!

5 QED (QCD) example (II)

5.1 two fermions and two photons

$$
\begin{aligned}
A_{+-\lambda_3\lambda_4} &= \frac{\langle 2^-|\not\epsilon^{\lambda_4}(k_4,q_4)(\not p_1+\not k_3)\not\epsilon^{\lambda_3}(k_3,q_3)|1^-\rangle}{s_{13}} \\
&+ \frac{\langle 2^-|\not\epsilon^{\lambda_3}(k_3,q_3)(\not p_1+\not k_4)\not\epsilon^{\lambda_4}(k_4,q_4)|1^-\rangle}{s_{14}} \qquad (95)
\end{aligned}
$$

For $\lambda_3=\lambda_4=-$, this vanishes if we take $q_3=q_4=p_1$.
For $\lambda_3=\lambda_4=+$, this vanishes if we take $q_3=q_4=p_2$.

$$
\begin{aligned}
A_{+-+-} &= \frac{\langle 2^-|\not\epsilon^{-}(k_4,q_4)(\not p_1+\not k_3)\not\epsilon^{+}(k_3,p_2)|1^-\rangle}{s_{13}} \\
&= \frac{\sqrt{2}}{[4q_4]}\langle 24\rangle\langle q_4^+|(\not p_1+\not k_3)|2^+\rangle[31]\frac{\sqrt{2}}{\langle 23\rangle}\frac{1}{s_{13}} \qquad (96)
\end{aligned}
$$

The choice $q_4=k_3$ leads to

$$
\begin{aligned}
A_{+-+-} &= \frac{\sqrt{2}}{[43]}\langle 24\rangle\langle 3^+|\not p_1|2^+\rangle[31]\frac{\sqrt{2}}{\langle 23\rangle}\frac{1}{s_{13}} = 2\frac{\langle 24\rangle[31]\langle 12\rangle[31]}{[43]\langle 23\rangle s_{13}} \\
&= 2\frac{\langle 24\rangle[31]\langle 12\rangle[31]}{[43]\langle 23\rangle\langle 13\rangle[31]} = 2\frac{\langle 24\rangle[34]\langle 42\rangle}{[34]\langle 23\rangle\langle 13\rangle} = -2\frac{\langle 24\rangle^2}{\langle 13\rangle\langle 23\rangle} \qquad (97)
\end{aligned}
$$

or

$$
\begin{aligned}
A_{+-+-} &= 2\frac{\langle 24\rangle[31]\langle 12\rangle[31]}{[43]\langle 23\rangle s_{13}} = -2\frac{[13]^2\langle 24\rangle\langle 12\rangle}{\langle 23\rangle[34]s_{24}} \\
&= +2\frac{[13]^2\langle 24\rangle\langle 12\rangle}{\langle 21\rangle[14]\langle 24\rangle[42]} = 2\frac{[13]^2}{[14][24]} \qquad (98)
\end{aligned}
$$

By exchanging $3\leftrightarrow 4$, we obtain

$$
A_{+--+} = -2\frac{\langle 23\rangle^2}{\langle 14\rangle\langle 24\rangle} \qquad (99)
$$

Unpolarized, helicity-summed cross sections

$$
|A|^2 \propto \frac{|s_{14}|}{|s_{13}|} + \frac{|s_{13}|}{|s_{14}|} \qquad (100)
$$

For $e^-\gamma\to e^-\gamma$ or $e^+\gamma\to e^+\gamma$,

$s_{13} = s$, $s_{12} = t$, $s_{14} = u$

$$|A|^2 \propto \frac{s^2 + u^2}{|su|} \tag{101}$$

For $e^- e^+ \to \gamma\gamma$,
$s_{12} = s$, $s_{13} = t$, $s_{14} = u$

$$|A|^2 \propto \frac{t^2 + u^2}{|tu|} \tag{102}$$

5.2 $e^- e^+ \to q g \bar{q}$

Using Fierz identity, we find

$$A_5 = \frac{\langle 25\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\epsilon}^+(k_4, q)|3^-\rangle}{\sqrt{2} s_{34}} + \frac{[13]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\epsilon}^+(k_4, q)|5^+\rangle}{\sqrt{2} s_{45}} \tag{103}$$

Using (92), we can rewrite A_5 as

$$A_5 = \frac{\langle 25\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|q^+\rangle[43]}{s_{34}\langle q4\rangle} + \frac{[13]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)|4^-\rangle\langle q5\rangle}{s_{45}\langle q4\rangle} \tag{104}$$

Choosing $q = k_5$, we remove the second term and get

$$\begin{aligned} A_5 &= -\frac{\langle 25\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|5^+\rangle[43]}{s_{34}\langle 45\rangle} \\ &= \frac{\langle 25\rangle\langle 1^+|(\not{k}_1 + \not{k}_2 + \not{k}_5)|5^+\rangle[43]}{\langle 12\rangle[21]\langle 34\rangle[43]\langle 45\rangle} \\ &= \frac{\langle 25\rangle[12]\langle 25\rangle[43]}{\langle 12\rangle[21]\langle 34\rangle[43]\langle 45\rangle} \\ &= -\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle} \end{aligned} \tag{105}$$

where we used $\sum_{i=1}^{5} k_i = 0$, Dirac eqs. and (53).

Note that

$$\frac{\langle 25\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle} = \frac{\langle 35\rangle}{\langle 34\rangle\langle 45\rangle} \frac{\langle 25\rangle^2}{\langle 12\rangle\langle 35\rangle} \tag{106}$$

6 Gluon amplitudes

- "Color" factors are separated (and ordered).
- Gluon vertices include $\epsilon \cdot \epsilon$.
- Since $\epsilon^+(k_1, q) \cdot \epsilon^+(k_2, q) = 0$, $A(+ + \cdots +) = 0$ can be shown.
- Using $\epsilon^-(k_1, q) \cdot \epsilon^+(k_2, k_1) = 0$, $A(- + \cdots +) = 0$ is also shown.

6.1 4-point amplitude

6.1.1 an amplitude

- Calculate $A_4(1^-2^-3^+4^+)$.
- Choose $q_1 = q_2 = k_3$, $q_3 = q_4 = k_2$, only nonvanishing $\epsilon \cdot \epsilon$ is $\epsilon^-(k_1, k_3) \cdot \epsilon^+(k_4, k_2) = \frac{\langle 21\rangle[43]}{\langle 24\rangle[31]}$.
- only s_{12} channel contributes

$$
\begin{aligned}
& A_4(1^-2^-3^+4^+) \\
= \; & \left(\frac{i}{\sqrt{2}}\right)^2 [-2k_1 \cdot \epsilon^-(k_2, k_3)\epsilon^-_\mu(k_1, k_3)] \frac{-ig^{\mu\nu}}{s_{12}} [2k_4 \cdot \epsilon^+(k_3, k_2)\epsilon^+_\nu(k_4, k_2)] \\
= \; & \frac{-2i}{s_{12}} \; k_1 \cdot \epsilon^-(k_2, k_3) \; k_4 \cdot \epsilon^+(k_3, k_2) \; \epsilon^-(k_1, k_3) \cdot \epsilon^+(k_4, k_2) \\
= \; & \frac{-2i}{s_{12}} \frac{[31]\langle 12\rangle}{\sqrt{2}[23]} \frac{\langle 24\rangle[43]}{\sqrt{2}\langle 23\rangle} \frac{\langle 21\rangle[43]}{\langle 24\rangle[31]} = -i \frac{\langle 12\rangle^2[34]^2}{s_{12}s_{23}} \\
= \; & i \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \qquad (107)
\end{aligned}
$$

Using cyclic symmetry, we get

$$
A_4(1^+2^-3^-4^+) = i \frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \qquad (108)
$$

6.1.2 decoupling identity

We consider $SU(N)$ gauge theory, not $U(N)$ gauge theory. The vertices come from the commutater in the field strength, thus if one of the generator T^a is replaced by unity, the amplitude (possibly thought as in $U(N)$ theory) must vanish.

The full amplitude looks like

$$A = g^2 \sum_{noncyclic\ perms} \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})A(1234) \tag{109}$$

(which can be generalized to n-point amplitudes)

Noncyclic means that, roughly speaking, 1 is fixed (or permutation $\sigma \in S_n/Z_n$), because of the cyclicity $A(1234) = A(4123)$.

Turning to the discussion, the following holds:

$$\begin{aligned} 0 &= \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3})[A(1234) + A(1243) + A(1423)] \\ &+ \mathrm{Tr}(T^{a_1}T^{a_3}T^{a_2})[A(1324) + A(1342) + A(1432)] \end{aligned} \tag{110}$$

Therefore we find

$$A(1234) + A(1243) + A(1423) = A(1324) + A(1342) + A(1432) = 0 \tag{111}$$

and then

$$A(1^-2^+3^-4^+) = -A(1^-2^+4^+3^-) - A(1^-4^+2^+3^-) \tag{112}$$

$$\begin{aligned} A(1^-2^+3^-4^+) &= -A(1^-2^+4^+3^-) - A(1^-4^+2^+3^-) \\ &= -A(3^-1^-2^+4^+) - A(3^-1^-4^+2^+) \\ &= -i\left[\frac{\langle 31\rangle^4}{\langle 31\rangle\langle 12\rangle\langle 24\rangle\langle 43\rangle} + \frac{\langle 31\rangle^4}{\langle 31\rangle\langle 14\rangle\langle 42\rangle\langle 23\rangle}\right] \\ &= -i\frac{\langle 13\rangle^3}{\langle 24\rangle}\left[\frac{1}{\langle 12\rangle\langle 34\rangle} + \frac{1}{\langle 14\rangle\langle 23\rangle}\right] \\ &= i\frac{\langle 13\rangle^3}{\langle 24\rangle}\frac{\langle 14\rangle\langle 23\rangle + \langle 12\rangle\langle 34\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\ &= i\frac{\langle 13\rangle^3}{\langle 24\rangle}\frac{-\langle 13\rangle\langle 42\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = i\frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \end{aligned} \tag{113}$$

where we used Schouten identity (30).

6.1.3 cross section

$$\sum_{colors} |A|^2 = 2N^2(N^2-1)g^4(|A(1234)|^2 + |A(1342)|^2 + |A(1423)|^2) \tag{114}$$

because of (6.1.2) and the reflection symmetry[1] $A(1234) = A(4321)$. Then

$$\sum_{colors} |A|^2(1^-2^-3^+4^+) = 2N^2(N^2-1)g^4 s^4 \left(\frac{1}{s^2u^2} + \frac{1}{t^2s^2} + \frac{1}{u^2t^2}\right) \tag{115}$$

[1] $A(12\cdots n) = (-1)^n A(n\cdots 21)$ in general

and summing over helicities, we find

$$\sum_{\substack{colors \\ helicities}} |A|^2 = 4N^2(N^2-1)g^4(s^4+t^4+u^4)\left(\frac{1}{s^2u^2}+\frac{1}{t^2s^2}+\frac{1}{u^2t^2}\right) \quad (116)$$

(for averaging over the initial colors we divide this by $4(N^2-1)^2$)

Column

Fun with Mandelstam!

$s+t+u=0$, okey?

$0=(s+t+u)^2=s^2+t^2+u^2+2(st+tu+us)$

$(s^2+t^2+u^2)^2 = s^4+t^4+u^4+2(s^2t^2+t^2u^2+u^2s^2) = 4(s^2t^2+t^2u^2+u^2s^2)+8(s^2tu+t^2us+u^2st) = 4(s^2t^2+t^2u^2+u^2s^2)$

$(s^4+t^4+u^4)(s^2+t^2+u^2) = -4(st+tu+us)(s^2t^2+t^2u^2+u^2s^2) = -4(st+tu+us)(s^2t^2+t^2u^2+u^2s^2-(s^2tu+t^2us+u^2st)) = -4(s^3t^3+t^3u^3+u^3s^3-3s^2t^2u^2)$

then

$(s^4+t^4+u^4)\left(\frac{1}{s^2u^2}+\frac{1}{t^2s^2}+\frac{1}{u^2t^2}\right) = 4\left(3-\frac{tu}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\right)$

7 Five-parton amplitude

- Calculate $A_5(1_{\bar{q}}^- 2_q^+ 3^- 4^+ 5^+)$
- Use $\epsilon^-(k_3, k_2)$, $\epsilon^+(k_4, k_1)$, $\epsilon^+(k_5, k_1)$ to apply (93)

$$A_5(1_{\bar{q}}^- 2_q^+ 3^- 4^+ 5^+) = A^{(a)} + A^{(b)} \tag{117}$$

$$\begin{aligned} A^{(a)} &= -\frac{i}{\sqrt{2}} \frac{\langle 2^+|(\not{k}_3 - \not{k}_4 - \not{k}_5)|1^+\rangle}{s_{12}s_{45}} \times \\ &\quad [\epsilon^-(k_3, k_2) \cdot \epsilon^+(k_5, k_1)\epsilon^+(k_4, k_1) \cdot k_5 \\ &\quad -\epsilon^-(k_3, k_2) \cdot \epsilon^+(k_4, k_1)\epsilon^+(k_5, k_1) \cdot k_4] \end{aligned} \tag{118}$$

$$\begin{aligned} A^{(b)} &= -\frac{i}{\sqrt{2}s_{12}s_{34}} \{\langle 2^+|(\not{k}_3 + \not{k}_4 - \not{k}_5)|1^+\rangle \times \\ &\quad [\frac{1}{2}\epsilon^-(k_3, k_2) \cdot \epsilon^+(k_4, k_1)\epsilon^+(k_5, k_1) \cdot (k_3 - k_4) \\ &\quad -\epsilon^-(k_3, k_2) \cdot \epsilon^+(k_5, k_1)\epsilon^+(k_4, k_1) \cdot k_3] \\ &\quad -\langle 2^+|(\not{k}_3 - \not{k}_4)|1^+\rangle \epsilon^-(k_3, k_2) \cdot \epsilon^+(k_4, k_1)\epsilon^+(k_5, k_1) \cdot (k_3 + k_4)\} \end{aligned} \tag{119}$$

$$\begin{aligned} A^{(a)} &= -i\frac{[23]\langle 31\rangle}{s_{12}s_{45}} \left[-\frac{[25]\langle 13\rangle}{[23]\langle 15\rangle}\frac{\langle 15\rangle[54]}{\langle 14\rangle} + \frac{[24]\langle 13\rangle}{[23]\langle 14\rangle}\frac{\langle 14\rangle[45]}{\langle 15\rangle} \right] \\ &= i\frac{[23]\langle 13\rangle^2[45]}{s_{12}s_{45}[23]\langle 14\rangle\langle 15\rangle} [-\langle 15\rangle[52] - \langle 14\rangle[42]] \\ &= -i\frac{[23]\langle 13\rangle^3[45]}{s_{12}s_{45}\langle 14\rangle\langle 15\rangle} = i\frac{[23]\langle 13\rangle^3}{s_{12}\langle 14\rangle\langle 15\rangle\langle 45\rangle} \end{aligned} \tag{120}$$

$$A^{(b)} = i\frac{[25]\langle 13\rangle^3[34]}{s_{12}s_{34}\langle 14\rangle\langle 15\rangle} = -i\frac{[25]\langle 13\rangle^3}{s_{12}\langle 14\rangle\langle 15\rangle\langle 34\rangle} \qquad (exercise!) \tag{121}$$

$$\begin{aligned} A_5(1_{\bar{q}}^- 2_q^+ 3^- 4^+ 5^+) &= A^{(a)} + A^{(b)} \\ &= -i\frac{\langle 13\rangle^3(-[23]\langle 34\rangle - [25]\langle 54\rangle)}{s_{12}\langle 14\rangle\langle 15\rangle\langle 34\rangle\langle 45\rangle} \\ &= -i\frac{\langle 13\rangle^3[21]\langle 14\rangle}{s_{12}\langle 14\rangle\langle 15\rangle\langle 34\rangle\langle 45\rangle} \\ &= i\frac{\langle 13\rangle^3\langle 23\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \end{aligned} \tag{122}$$

Here we used momentum conservation, (86) and (91).

8 General gluon amplitudes

8.1 Recursive relations

Berends-Giele current $J^\mu(1\cdots n)$

$$J^\mu(1\cdots n) = -\frac{i}{K_{1,n}^2}[\sum_{j=1}^{n-1} V_3^{\mu\nu\lambda}(K_{1,j}, K_{j+1,n}) J_\nu(1,\ldots,j) J_\lambda(j+1,\ldots,n) + \sum_{j=1}^{n-2}\sum_{\ell=j+1}^{n-1} V_4^{\mu\nu\lambda\rho} J_\nu(1,\ldots,j) \times J_\lambda(j+1,\ldots,\ell) J_\rho(\ell+1,\ldots,n)] \tag{123}$$

where V are the color-ordered gluon self-interactions

$$V_3^{\mu\nu\lambda}(P,Q) = \frac{i}{\sqrt{2}}[\eta^{\nu\lambda}(P-Q)^\mu + 2\eta^{\lambda\mu}Q^\nu - 2\eta^{\mu\nu}Q^\lambda] \tag{124}$$

$$V_4^{\mu\nu\lambda\rho} = \frac{i}{2}[2\eta^{\mu\lambda}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\rho}\eta^{\nu\lambda}] \tag{125}$$

and

$$K_{i,j} \equiv k_i + k_{i+1} + \cdots + k_j \tag{126}$$

- decoupling identity
  $J^\mu(123\cdots n) + J^\mu(213\cdots n) + \cdots + J^\mu(23\cdots n1) = 0$
- reflection identity
  $J^\mu(123\cdots n) = (-1)^{n+1} J^\mu(n\cdots 321)$
- conservation
  $K_{1,n}^\mu J_\mu(12\cdots n) = 0$

8.2 example: $J^\mu(+\cdots+)$

By explicit calculation, one can get

$$J^\mu(1^+2^+3^+4^+) = \frac{\langle q^-|\gamma^\mu \not{K}_{1,4}|q^+\rangle}{\sqrt{2}\langle q1\rangle\langle\langle 1\cdots 4\rangle\rangle\langle 4q\rangle} \tag{127}$$

where $\langle\langle 1\cdots n\rangle\rangle \equiv \langle 12\rangle\langle 23\rangle\cdots\langle (n-1)\, n\rangle$.

Ansatz

$$J^\mu(1^+\cdots n^+) = \frac{\langle q^-|\gamma^\mu \not{K}_{1,n}|q^+\rangle}{\sqrt{2}\langle q1\rangle\langle\langle 1\cdots n\rangle\rangle\langle nq\rangle} \tag{128}$$

(Reflection and Conservation are trivially satisfied; ?Decoupling?)

· Note that

$$J^\mu(1^+) = \frac{\langle q^-|\gamma^\mu \not{k}_1|q^+\rangle}{\sqrt{2}\langle q1\rangle\langle 1q\rangle} = \frac{\langle q^-|\gamma_\mu|1^-\rangle}{\sqrt{2}\langle q1\rangle} = \epsilon^{+\mu}(k_1, q) \qquad (129)$$

In the right-hand side of (123), only the second and third term in V_3 contribute.

$$\begin{aligned} J^\mu(1^+2^+\cdots n^+) &= \frac{1}{\sqrt{2}K_{1,n}^2\langle q1\rangle\langle\langle 1\cdots n\rangle\rangle\langle nq\rangle}\sum_{j=1}^{n-1}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle} \\ &\times(\langle q^-|\gamma^\mu \not{K}_{j+1,n}|q^+\rangle K^\nu_{j+1,n}\langle q^-|\gamma_\nu \not{K}_{1,j}|q^+\rangle \\ &-\langle q^-|\gamma^\mu \not{K}_{1,j}|q^+\rangle K^\nu_{1,j}\langle q^-|\gamma_\nu \not{K}_{j+1,n}|q^+\rangle) \end{aligned} \qquad (130)$$

Note that

$$\langle q^-|\gamma_\mu\gamma_\nu|q^+\rangle = -\langle q^-|\gamma_\nu\gamma_\mu|q^+\rangle + 2\eta_{\mu\nu}\langle q^-|q^+\rangle = -\langle q^-|\gamma_\nu\gamma_\mu|q^+\rangle\,, \qquad (131)$$

and $K_{1,j} + K_{j+1,n} = K_{1,n}$.

We find

$$\begin{aligned} J^\mu(1^+2^+\cdots n^+) &= \frac{\langle q^-|\gamma^\mu \not{K}_{1,n}|q^+\rangle}{\sqrt{2}K_{1,n}^2\langle q1\rangle\langle\langle 1\cdots n\rangle\rangle\langle nq\rangle} \\ &\times\sum_{j=1}^{n-1}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle}\langle q^-|\not{K}_{j+1,n}\not{K}_{1,j}|q^+\rangle \\ &= \frac{\langle q^-|\gamma^\mu \not{K}_{1,n}|q^+\rangle}{\sqrt{2}K_{1,n}^2\langle q1\rangle\langle\langle 1\cdots n\rangle\rangle\langle nq\rangle} \\ &\times\sum_{j=1}^{n-1}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle}\langle q^-|\not{K}_{j+1,n}\not{K}_{1,n}|q^+\rangle \\ &= -\frac{\langle q^-|\gamma^\mu \not{K}_{1,n}|q^+\rangle}{\sqrt{2}K_{1,n}^2\langle q1\rangle\langle\langle 1\cdots n\rangle\rangle\langle nq\rangle} \\ &\times\sum_{j=1}^{n-1}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle}\langle q^-|\not{K}_{1,j}\not{K}_{1,n}|q^+\rangle \end{aligned} \qquad (132)$$

for

$$\langle q^-|\not{K}\not{K}|q^+\rangle = K^2\langle q^-|q^+\rangle = 0\,. \qquad (133)$$

To proceed, we use

$$\sum_{j=1}^{n-1}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle}\langle q^-|\not{K}_{1,j} = \sum_{j=1}^{n-1}\sum_{\ell=1}^{j}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle}\langle q^-|\not{k}_\ell$$

$$= \sum_{\ell=1}^{n}\sum_{j=\ell}^{n-1}\frac{\langle j\,(j+1)\rangle}{\langle jq\rangle\langle q\,(j+1)\rangle}\langle q^-|\not{k}_\ell = \sum_{\ell=1}^{n}\frac{\langle \ell\, n\rangle}{\langle \ell\, q\rangle\langle q\, n\rangle}\langle q^-|\not{k}_\ell$$

$$= \sum_{\ell=1}^{n}\frac{1}{\langle q\, n\rangle}\langle n^-|\not{k}_\ell = \frac{1}{\langle q\, n\rangle}\langle n^-|\not{K}_{1,n} \tag{134}$$

where "eikonal" identity (31), ('open') Schouten identity

$$\langle \ell n\rangle\langle q^-| + \langle q\ell\rangle\langle n^-| + \langle nq\rangle\langle \ell^-| = 0 \tag{135}$$

and Dirac equations $\langle \ell^-|\not{k}_\ell = 0$ are used.

Then (128) is shown.

Show $A(+\cdots+) = A(-+\cdots+) = 0$ by using $J(+\cdots+)$.

8.3 $J^\mu(-+\cdots+)$ and MHV amplitude

$$J^\mu(1^-2^+\cdots n^+) = \frac{\langle 1^-|\gamma^\mu\not{K}_{2,n}|1^+\rangle}{\sqrt{2}\langle\langle 1\cdots n\rangle\rangle\langle n1\rangle}\sum_{m=3}^{n}\frac{\langle 1^-|\not{k}_m\not{K}_{1,m}|1^+\rangle}{K_{1,m-1}^2K_{1,m}^2} \tag{136}$$

where $q_1 = k_2$, $q_2 = \cdots = q_n = k_1$. Show this!!

$$A(1^-2^+\cdots n^+(n{+}1)^-) = -i\frac{\langle n^+|\gamma_\mu|(n+1)^+\rangle}{\sqrt{2}[n\,(n+1)]}\frac{\langle 1^-|\gamma^\mu\not{K}_{1,n}|1^+\rangle}{\sqrt{2}\langle\langle 1\cdots n\rangle\rangle\langle n1\rangle}\frac{\langle 1^-|\not{k}_n\not{K}_{1,n}|1^+\rangle}{K_{1,n-1}^2} \tag{137}$$

where we choose $q_{n+1} = k_n$, and note that $\not{K}_{2,n}|1^+\rangle = \not{K}_{1,n}|1^+\rangle$. Moreover we note that $k_{n+1} = -K_{1,n}$ is null.

Fierz identity leads to

$$\langle n^+|\gamma_\mu|(n+1)^+\rangle\langle 1^-|\gamma^\mu\not{K}_{1,n}|1^+\rangle = 2\langle 1\,(n+1)\rangle\langle n^+|\not{K}_{1,n}|1^+\rangle \tag{138}$$

while

$$\langle n^+|\not{K}_{1,n}|1^+\rangle = -\langle n^+|\not{k}_{n+1}|1^+\rangle = -[n\,(n+1)]\langle (n+1)\,1\rangle \tag{139}$$

and

$$\langle 1^-|\not{k}_n\not{K}_{1,n}|1^+\rangle = -\langle 1^-|\not{k}_n\not{k}_{n+1}|1^+\rangle = -\langle 1n\rangle[n\,(n+1)]\langle (n+1)\,1\rangle \tag{140}$$

Therefore

$$A(1^-2^+\cdots n^+(n+1)^-) = i\frac{\langle 1\,(n+1)\rangle[n\,(n+1)]\langle (n+1)\,1\rangle\langle (n+1)\,1\rangle}{\langle\langle 1\cdots n\rangle\rangle s_{n,n+1}}$$

$$= i\frac{\langle 1\,(n+1)\rangle^4}{\langle\langle 1\cdots (n+1)\rangle\rangle\langle (n+1)\,1\rangle} \tag{141}$$

9 'Supersymmetry'

9.1 'susy' generator

$Q(\eta) = \bar{\eta}^\alpha Q_\alpha$ is bosonic.

$$[Q(\eta), G^\pm(k)] = \pm\Gamma^\pm(k,\eta)\Lambda^\pm(k)\,, \quad [Q(\eta), \Lambda^\pm(k)] = \mp\Gamma^\mp(k,\eta)G^\pm(k) \quad (142)$$

where $\Gamma(k,\eta)$ is linear in η and fermionic.

The Jacobi identity

$$0 = [[Q(\eta), Q(\zeta)], \Phi(k)] + [[Q(\zeta), \Phi(k)], Q(\eta)] + [[\Phi(k), Q(\eta)], Q(\zeta)] \quad (143)$$

leads to a possible choice $\Gamma^+(k,q) = \theta[qk]$ and $\Gamma^-(k,q) = \theta\langle qk\rangle$ where θ is a fermionic parameter ($\bar{\eta}(q) \sim \theta\overline{\mathcal{U}_+}(q)$).

9.2 'Supersymmetry' Ward identity

$$0 = \langle 0|[Q, \Phi_1\Phi_2\cdots\Phi_n]|0\rangle = \sum_{i=1}^{n}\langle 0|\Phi_1\cdots[Q,\Phi_i]\cdots\Phi_n|0\rangle \quad (144)$$

9.2.1 all helicities positive

$$\begin{aligned} 0 &= \langle 0|[Q(\eta(q)), \Lambda_1^+ G_2^+ \cdots G_n^+]|0\rangle \\ &= -\Gamma^-(k_1,q)A_n(G_1^+, G_2^+, \cdots, G_n^+) + \Gamma^+(k_2,q)A_n(\Lambda_1^+, \Lambda_2^+, G_3^+, \cdots, G_n^+) \\ &\quad + \cdots + \Gamma^+(k_n,q)A_n(\Lambda_1^+, G_2^+, \cdots, \Lambda_n^+) \end{aligned} \quad (145)$$

Helicity conservation of 'gluino' implies that the fermionic amplitudes vanish, thus $A_n(G_1^+, G_2^+, \cdots, G_n^+) = 0$.

9.2.2 only one negative helicity

$$\begin{aligned} 0 &= \langle 0|[Q(\eta(q)), \Lambda_1^+ G_2^- G_3^+ \cdots G_n^+]|0\rangle \\ &= -\Gamma^-(k_1,q)A_n(G_1^+, G_2^-, G_3^+, \cdots, G_n^+) - \Gamma^-(k_2,q)A_n(\Lambda_1^+, \Lambda_2^-, G_3^+, \cdots, G_n^+) \\ &\quad +0 + \cdots + 0 \end{aligned} \quad (146)$$

If we choose $q = k_1$, we get $A_n(\Lambda_1^+, \Lambda_2^-, G_3^+, \cdots, G_n^+) = 0$. If we choose $q = k_2$, we get $A_n(G_1^+, G_2^-, G_3^+, \cdots, G_n^+) = 0$.

9.2.3 two negative helicity

$$
\begin{aligned}
0 &= \langle 0|[Q(\eta(q)), G_1^- G_2^- \Lambda_3^+ G_4^+ \cdots G_n^+]|0\rangle \\
&= \Gamma^-(k_1,q) A_n(\Lambda_1^-, G_2^-, \Lambda_3^+, \cdots, G_n^+) + \Gamma^-(k_2,q) A_n(G_1^-, \Lambda_2^-, \Lambda_3^+, \cdots, G_n^+) \\
&\quad -\Gamma^-(k_3,q) A_n(G_1^-, G_2^-, G_3^+, \cdots, G_n^+) \qquad (147)
\end{aligned}
$$

If we choose $q = k_1$, we get

$$
A_n(G_1^-, G_2^-, G_3^+, \cdots, G_n^+) = \frac{\langle 12\rangle}{\langle 13\rangle} A_n(G_1^-, \Lambda_2^-, \Lambda_3^+, G_4^+, \cdots, G_n^+) \qquad (148)
$$

References

[1] L. Dixon, hep-ph/9601359, and his recent talks which can be found in various conferences.

[2] M. Mangano and S. Parke, Phys. Rep. 200 (1991) 301. The preprint of this can be obtained from Parke's page.

[3] D. A. Kosower, "*N=4 Supersymmetric Gauge Theory, Twistor Space, and Dualities*", Saclay Lectures (Fall 2004).