曲率の計算：球対称な場合

白石　清（山口大院理工）

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概 要

Einstein 方程式の球対称な解を求めるために。

1 時空

1.1 $(N+1)$ 次元時空の metric

球対称

$$ds^2 = -e^{-2\delta}\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_{N-1}^2 \tag{1}$$

Δ, δ は r のみの関数と仮定する。

1.2 vielbein

$$e^0 = e^{-\delta}\sqrt{\Delta}dt \tag{2}$$

$$e^1 = \frac{1}{\sqrt{\Delta}}dr \tag{3}$$

$$e^A = r\tilde{e}^A \tag{4}$$

ここで

$$d\tilde{e}^A + \tilde{\omega}^A{}_B \wedge \tilde{e}^B = 0 \tag{5}$$

$A, B = 2, \ldots, N$ とする。

$$de^0 = \left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right) dr \wedge e^0 = \sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^1 \wedge e^0 \tag{6}$$

$$de^1 = 0 \tag{7}$$

$$de^A = \frac{1}{r}dr \wedge e^A + r d\tilde{e}^A = \frac{\sqrt{\Delta}}{r}e^1 \wedge e^A - \tilde{\omega}^A{}_B \wedge e^B \tag{8}$$

1.3 spin connection

$$\omega^0{}_1 = \sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right) e^0 \tag{9}$$

$$\omega^A{}_1 = \frac{\sqrt{\Delta}}{r} e^A \tag{10}$$

$$\omega^A{}_B = \tilde{\omega}^A{}_B \tag{11}$$

1.4 curvature 2-form

$$\Theta^0{}_1 = d\omega^0{}_1$$
$$= -\left\{\sqrt{\Delta}\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]' + \left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]^2\right\} e^0 \wedge e^1 \tag{12}$$

$$\begin{aligned}\Theta^A{}_1 &= d\omega^A{}_1 + \omega^A{}_B \wedge \omega^B{}_1 \\ &= -\left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} e^A \wedge e^1\end{aligned} \tag{13}$$

$$\begin{aligned}\Theta^A{}_B &= d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B + \omega^A{}_1 \wedge \omega^1{}_B \\ &= \left\{\frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} e^A \wedge e^B\end{aligned} \tag{14}$$

$$\begin{aligned}\Theta^0{}_A &= \omega^0{}_1 \wedge \omega^1{}_A \\ &= -\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]\left[\frac{\sqrt{\Delta}}{r}\right] e^0 \wedge e^A\end{aligned} \tag{15}$$

1.5 Riemann tensor

$$R^{01}{}_{01} = -\left\{\sqrt{\Delta}\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]' + \left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]^2\right\} \tag{16}$$

$$R^{A1}{}_{B1} = -\left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\}\delta^A_B \tag{17}$$

$$R^{AB}{}_{CD} = \left\{\frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\}\left(\delta^A_C\delta^B_D - \delta^A_D\delta^B_C\right) \tag{18}$$

$$R^{0A}{}_{0B} = -\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]\left[\frac{\sqrt{\Delta}}{r}\right]\delta^A_B \tag{19}$$

1.6 Ricci tensor

$$
\begin{aligned}
R^0_0 &= R^{01}{}_{01} + R^{0A}{}_{0A} \\
&= -\left\{\sqrt{\Delta}\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]' + \left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]^2\right\} \\
&- (N-1)\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]\left[\frac{\sqrt{\Delta}}{r}\right] \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
R^1_1 &= R^{01}{}_{01} + R^{A1}{}_{A1} \\
&= -\left\{\sqrt{\Delta}\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]' + \left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]^2\right\} \\
&- (N-1)\left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
R^A_B &= R^{0A}{}_{0B} + R^{A1}{}_{B1} + R^{AC}{}_{BC} \\
&= \left(-\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]\left[\frac{\sqrt{\Delta}}{r}\right]\right. \\
&- \left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \\
&+ \left.(N-2)\left\{\frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\}\right)\delta^A_B \qquad (22)
\end{aligned}
$$

1.7 scalar curvature

$$\begin{aligned}
R &= -2\left\{\sqrt{\Delta}\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]' + \left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]^2\right\} \\
&- 2(N-1)\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]\left[\frac{\sqrt{\Delta}}{r}\right] \\
&- 2(N-1)\left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \\
&+ (N-1)(N-2)\left\{\frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \qquad (23)
\end{aligned}$$

1.8 Einstein tensor

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu \qquad (24)$$

$$\begin{aligned}
G^0_0 &= (N-1)\left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \\
&- \frac{(N-1)(N-2)}{2}\left\{\frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \qquad (25)
\end{aligned}$$

$$\begin{aligned}
G^0_0 - G^1_1 &= R^0_0 - R^1_1 \\
&= -(N-1)\left[\sqrt{\Delta}\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]\left[\frac{\sqrt{\Delta}}{r}\right] \\
&+ (N-1)\left\{\sqrt{\Delta}\left[\frac{\sqrt{\Delta}}{r}\right]' + \left[\frac{\sqrt{\Delta}}{r}\right]^2\right\} \qquad (26)
\end{aligned}$$

1.9　使う式

$$G_0^0 = \frac{N-1}{2r}\Delta' + \frac{(N-1)(N-2)}{2}\frac{\Delta - 1}{r^2} \tag{27}$$

$$G_0^0 - G_1^1 = \Delta\frac{N-1}{r}\delta' \tag{28}$$

2 Einstein equation

$$G^\mu_\nu = R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu = 8\pi G T^\mu_\nu \tag{29}$$

$(N+1)$ 次元では

$$R^\mu_\nu = 8\pi G\left(T^\mu_\nu - \frac{1}{N-1}T\delta^\mu_\nu\right) \tag{30}$$

完全流体 (perfect fluid) を仮定すると

$$T^\mu_\nu = diag.(-\rho, P, P, \cdots, P) \tag{31}$$

とすると

$$T = T^\mu_\mu = -\rho + NP \tag{32}$$

$$G^0_0 = -8\pi G\rho \tag{33}$$

$$G^0_0 - G^1_1 = -8\pi G(\rho + P) \tag{34}$$

$$\tag{35}$$

$$\frac{N-1}{2r}\Delta' + \frac{(N-1)(N-2)}{2}\frac{\Delta - 1}{r^2} = -8\pi G\rho \tag{36}$$

$$\Delta\left[\frac{N-1}{r}\delta'\right] = -8\pi G(\rho + P) \tag{37}$$

3 conservation

$$\nabla_\lambda T^{\mu\nu} = \partial_\lambda T^{\mu\nu} + \Gamma^\mu_{\lambda\sigma} T^{\sigma\nu} + \Gamma^\nu_{\lambda\sigma} T^{\mu\sigma} \tag{38}$$

これより

$$\begin{aligned}\nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} \\ &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} T^{\mu\nu} \right) + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} \end{aligned} \tag{39}$$

ここで

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} = \frac{1}{2} g^{\lambda\sigma} \partial_\mu g_{\lambda\sigma} = \Gamma^\lambda_{\mu\lambda} \tag{40}$$

を使った。

$$T^{tt} = \frac{1}{e^{-2\delta}\Delta} \rho \tag{41}$$

$$T^{rr} = \Delta P \tag{42}$$

$$T^{ij} = \frac{1}{r^2} \tilde{g}^{ij} P \tag{43}$$

$$\tag{44}$$

および

$$\Gamma^r_{tt} = e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \tag{45}$$

$$\Gamma^r_{rr} = -\frac{1}{2} \frac{\Delta'}{\Delta} \tag{46}$$

$$\Gamma^r_{ij} = -\Delta r \tilde{g}_{ij} \tag{47}$$

$$\tag{48}$$

などを使うと，保存の式，あるいは力学的平衡の式

$$\nabla_\mu T^{\mu r} = 0 \tag{49}$$

は次のようになる。

$$P' + \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) (\rho + P) = 0 \tag{50}$$

4　重力平衡の式

アインシュタイン方程式とあわせると，

$$-P' = \frac{8\pi G}{(N-1)\Delta} r \left(\frac{(N-1)(N-2)}{16\pi G} \frac{1-\Delta}{r^2} + P \right) (\rho + P) \tag{51}$$

$$\Delta = 1 - \frac{16\pi G M_r}{(N-1)A_{N-1} r^{N-2}} \tag{52}$$

とおくと

$$-P' = \frac{8\pi G}{(N-1)\left(1 - \frac{16\pi G M_r}{(N-1)A_{N-1} r^{N-2}}\right)} r \left(\frac{(N-2)M_r}{A_{N-1} r^N} + P \right) (\rho + P) \tag{53}$$

ニュートン近似では

$$-P' = \frac{8\pi G (N-2) M_r}{(N-1)A_{N-1} r^{N-1}} \rho \tag{54}$$

を得る。

なお，

$$M_r(r) = A_{N-1} \int_0^r \rho(r) {r'}^{N-1} dr' \tag{55}$$

である。

$$A_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} \tag{56}$$

Appendix

$$ds^2 = -e^{-2\delta}\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_{N-1}^2 \tag{57}$$

Δ, δ は r のみの関数と仮定する。

$$d\Omega_{N-1}^2 = \tilde{g}_{ij} d\tilde{x}^i \tilde{x}^j \qquad (i, j = 2, \ldots, N) \tag{58}$$

$$\begin{aligned}
\Gamma^t_{rt} &= -\delta' + \frac{1}{2}\frac{\Delta'}{\Delta} && (59)\\
\Gamma^r_{tt} &= e^{-2\delta}\Delta^2\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right) && (60)\\
\Gamma^r_{rr} &= -\frac{1}{2}\frac{\Delta'}{\Delta} && (61)\\
\Gamma^r_{ij} &= -\Delta r \tilde{g}_{ij} && (62)\\
\Gamma^i_{rj} &= \frac{1}{r}\delta^i_j && (63)\\
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} && (64)
\end{aligned}$$

$$\begin{aligned}
\Gamma^\lambda_{t\lambda} &= 0 && (65)\\
\Gamma^\lambda_{r\lambda} &= \Gamma^t_{rt} + \Gamma^r_{rr} + \Gamma^k_{rk} = -\delta' + \frac{N-1}{r} && (66)\\
\Gamma^\lambda_{i\lambda} &= \tilde{\Gamma}^k_{ik} && (67)
\end{aligned}$$

$$R_{\sigma\nu} = \partial_\lambda \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\sigma\lambda} + \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\nu\sigma} - \Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\sigma\lambda} \tag{68}$$

$$\begin{aligned}
R_{tt} &= \partial_r \Gamma^r_{tt} - 0 + \Gamma^r_{tt}\Gamma^\lambda_{r\lambda} - 2\Gamma^r_{tt}\Gamma^t_{tr}\\
&= \left[e^{-2\delta}\Delta^2\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\right]'
\end{aligned}$$

$$
\begin{aligned}
&+ \quad e^{-2\delta}\Delta^2\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left[-\delta' + \frac{N-1}{r}\right] \\
&- \quad 2e^{-2\delta}\Delta^2\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 \\
&= \quad e^{-2\delta}\Delta^2\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)' \\
&+ \quad e^{-2\delta}\Delta^2\left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)\left[-\delta' + \frac{N-1}{r} + \frac{\Delta'}{\Delta}\right] \qquad (69)
\end{aligned}
$$

$$
\begin{aligned}
R_{rr} &= \partial_r\Gamma^r_{rr} - \partial_r\Gamma^\lambda_{r\lambda} + \Gamma^r_{rr}\Gamma^\lambda_{r\lambda} - \left(\Gamma^t_{rt}{}^2 + \Gamma^r_{rr}{}^2 + \Gamma^i_{rj}\Gamma^j_{ri}\right) \\
&= \left(-\frac{1}{2}\frac{\Delta'}{\Delta}\right)' - \left[-\delta' + \frac{N-1}{r}\right]' + \left(-\frac{1}{2}\frac{\Delta'}{\Delta}\right)\left[-\delta' + \frac{N-1}{r}\right] \\
&- \left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 - \left(-\frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 - \frac{N-1}{r^2} \\
&= -\left[-\delta' + \frac{N-1}{r} + \frac{1}{2}\frac{\Delta'}{\Delta}\right]' + \left(-\frac{1}{2}\frac{\Delta'}{\Delta}\right)\left[-\delta' + \frac{N-1}{r} + \frac{1}{2}\frac{\Delta'}{\Delta}\right] \\
&- \left(-\delta' + \frac{1}{2}\frac{\Delta'}{\Delta}\right)^2 - \frac{N-1}{r^2} \qquad (70)
\end{aligned}
$$

$$
\begin{aligned}
R_{ij} &= \partial_r\Gamma^r_{ij} + \partial_k\Gamma^k_{ij} - \partial_i\Gamma^\lambda_{j\lambda} + \Gamma^r_{ij}\Gamma^\lambda_{r\lambda} + \Gamma^k_{ij}\Gamma^\lambda_{k\lambda} - \left(2\Gamma^k_{ri}\Gamma^r_{jk} + \Gamma^k_{\ell i}\Gamma^\ell_{jk}\right) \\
&= [-\Delta r]'\,\tilde{g}_{ij} + \partial_k\tilde{\Gamma}^k_{ij} - \partial_i\tilde{\Gamma}^k_{jk} + [-\Delta r]\left[-\delta' + \frac{N-1}{r}\right]\tilde{g}_{ij} \\
&+ \tilde{\Gamma}^k_{ij}\tilde{\Gamma}^\ell_{k\ell} - 2\,[-\Delta]\,\tilde{g}_{ij} - \tilde{\Gamma}^k_{\ell i}\tilde{\Gamma}^\ell_{jk} \\
&= \tilde{R}_{ij} - \Delta r\left[-\delta' + \frac{N-2}{r} + \frac{\Delta'}{\Delta}\right]\tilde{g}_{ij} \qquad (71)
\end{aligned}
$$

参考文献

[1] ???