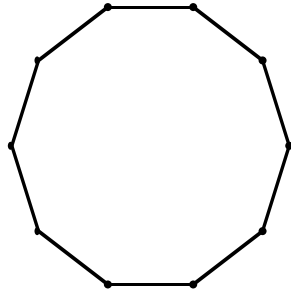


Divergences in QED on a Graph

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§ 1. INTRODUCTION

Dimensional Deconstruction (離散的な余次元空間のモデル)



$[4D]^N$ latticized 5D

より高次元を目指すには 点と辺のつながり方 はもっと複雑

グラフ理論

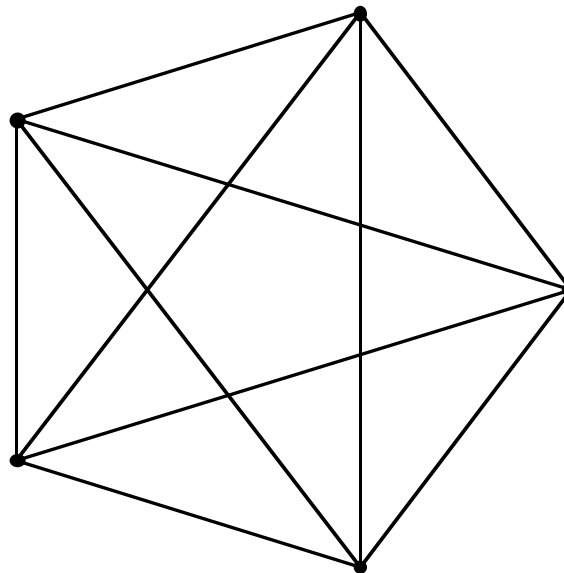
グラフ上の場の理論 (Deconstructionの一般化)

- ・ スカラー場のone-loop effective potentialは有限
離散的な幾何学の何がどのように効いているのか？

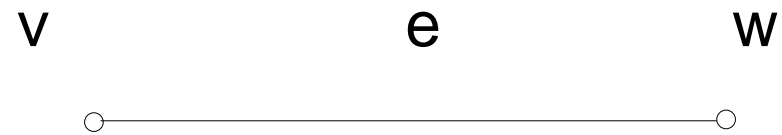
Spectral Graph Theory

§ 2. GRAPH THEORY

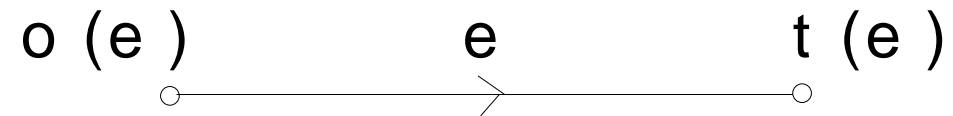
$G (V (G), E (G))$ $\#V(G) = p$: 位数
 $\#E(G) = q$: サイズ



$v \sim w$ (v と w は隣接)

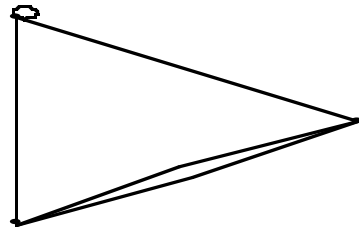


有向辺



$$e = [o(e), t(e)]$$

以下， G ：単純グラフ（ループや多重辺を含まない）



これは単純グラフではない

頂点 v の次数 $\deg(v)$: v を端点とする辺の数

隣接行列

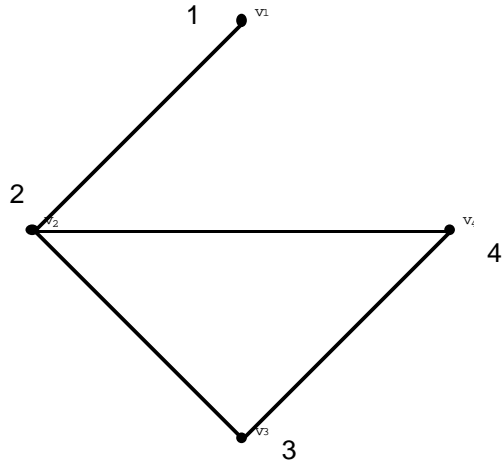
$$(A)_{vv'} = \begin{cases} 1 & \text{if } v \sim v' \\ 0 & \text{otherwise} \end{cases}$$

次数行列

$$(D)_{vv'} = \begin{cases} \deg(v) & v = v' \\ 0 & \text{otherwise} \end{cases}$$

ラプラシアン

D - A

例

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

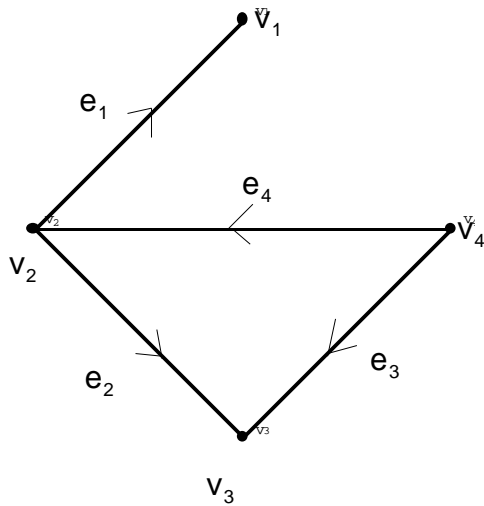
$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

有向グラフの接続行列

$$(E)_{ve} = \begin{cases} 1 & \text{if } v = o(e) \\ -1 & \text{if } v = t(e) \\ 0 & \text{otherwise} \end{cases}$$

例



$$E = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= E E^T$$

§ 3. FIELD THEORY ON A GRAPH (U(1))

$$L_V = - \frac{1}{4g^2} \sum_v F_v^\mu F_{v\mu} + \frac{1}{2} \sum_e f_e^2 \left| D^\mu U_e \right|^2$$

ここで $D^\mu U_e = \partial^\mu U_e - iA_{o(e)}^\mu U_e + iU_e A_{t(e)}^\mu$

$$U_e = \exp(-i \int_e A^\mu dx_\mu)$$

ゲージ変換 $A_v^\mu \rightarrow A_v^\mu + \partial_v^\mu \alpha_v$,

$$U_e \rightarrow W_{o(e)} U_e W_{t(e)}^\dagger \quad W_v = e^{i\alpha_v(x)}$$

$$(\partial_e^\mu + (d^\mu)_e), (d^\mu)_e = \partial_{t(e)}^\mu - \partial_{o(e)}^\mu$$

gauge fixing term

$$L_{\text{gf}} = - \int_V \frac{1}{2g^2} \left(\partial_\mu A_\nu - (gf)^2 (E)_\nu \right)^2$$

ghost term

$$L_{\text{ghost}} = \int_V \bar{c}_\nu \left[- \left(\partial_\nu^2 + (gf)^2 EE^T \right) c \right]_\nu$$

The gauge-fixed lagrangian

$$\begin{aligned}
 \mathcal{L}_V = & -\frac{1}{2g^2} \sum_{\nu} \left[\left(\sum_{\mu} \partial_{\nu} A_{\mu} \right)^2 - \left(1 - \frac{1}{\alpha} \right) \left(\sum_{\mu} \partial_{\mu} A_{\nu} \right)^2 \right] + \frac{1}{2} f^2 \sum_{\nu} A_{\nu}^{\mu} A_{\mu\nu} \\
 & + \frac{1}{2} \sum_{e} \left(\sum_{E} X_e \right)^2 - \frac{1}{2} (gf)^2 \sum_{e} X_e \tilde{X}_e \\
 & + \sum_{\nu} \bar{c}_{\nu} \left[-\left(\sum_{\nu} \partial_{\nu}^2 + (gf)^2 \right) c \right]_{\nu}
 \end{aligned}$$

$X_e = f \sum_{e} \partial_{\nu} A_{\nu}^{\mu} A_{\mu\nu}$, $\tilde{X}_e = \sum_{E} \partial_{\nu} A_{\nu}^{\mu} A_{\mu\nu}$, $\tilde{X}_e = E^T E$ **結局, \tilde{X}_e は Stueckelberg field.**

Feynman gauge ($\alpha = 1$) vector fields, scalar fields, ghost fields
 同じ mass spectrum

note

$$= E E^T$$

固有値0をもつ。(重複度は1)

$$\left\{ \begin{array}{ll} \text{zero固有値} & 1\text{個} \\ \text{non-zero固有値} & p - 1\text{個} \end{array} \right.$$

$$\sim = E^T E$$

と同じスペクトル + $q - p$ 個のzero固有値

$$\left\{ \begin{array}{ll} \text{zero固有値} & q - p + 1\text{個} \\ \text{non-zero固有値} & p - 1\text{個} \end{array} \right.$$

実際

ゲージ変換

$$e_e = (d_e + E_e^T) e_e$$

rank $E = p - 1$ よりゲージ変換の自由度は $p - 1$

$p - 1$ 個のスカラー場 gauged away

$q - p + 1$ 個のphysicalなmasslessスカラー場

physical massless scalar modes $\begin{pmatrix} (i) \\ 0 \end{pmatrix}$ ($i = 1, \dots, q - p + 1$)

ゲージ変換と直交

$$\begin{pmatrix} (i) \\ 0 \end{pmatrix} \begin{pmatrix} E^T \\ e \end{pmatrix} = 0$$

$$\begin{pmatrix} E \\ 0 \end{pmatrix} \begin{pmatrix} (i) \\ e \end{pmatrix} = 0$$

$\begin{pmatrix} (i) \\ 0 \end{pmatrix}$ は基本タイセット行列の行ベクトルで表される。

note基本タイセット行列 F_f

$$(F_f)_{ij} = \begin{cases} 1 & : L_i \text{ が辺 } e_j \text{ を正の向きに含む} \\ -1 & : L_i \text{ が辺 } e_j \text{ を負の向きに含む} \\ 0 & : \text{その他} \end{cases}$$

L_i : G の全域木 T に関する基本閉路

例

$$F_f = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 \end{array} \right]$$

基本閉路の数

$$= \#E(\bar{T})$$

$$= q - \#E(T)$$

$$= q - (p - 1)$$

$$= q - p + 1$$

$n(G)$: G の零度 (G の閉路階数)

$$\bullet \text{rank } F_f = n(G)$$

$$\bullet E F_f^T = 0$$

定理 (?)

$\tilde{A} = E^T E$ のゼロ固有値 (重複度 $n(G)$) に

属する固有ベクトルを y_0 とすると

y_0^T は基本タイセット行列の行ベクトル

) \tilde{A} は正規行列 ($\tilde{A} \tilde{A}^T = \tilde{A}^T \tilde{A}$)

代数的重複度 = 幾何学的重複度

ゼロ固有値に対して

$$E^T E y_0 = 0$$

$$E^T E Y = 0 : Y = (y_0 \ y_0 \ \cdots \ y_0)$$

n(G)個

$$Y \neq 0, E Y = 0, E^T E Y = 0 \text{ より}$$

$$E Y = 0$$

$$Y = F_f^T \quad (E F_f^T = 0)$$

$$y_0^T = f_f$$

§ 4. FERMIONS ON A GRAPH

$$\begin{aligned}
 L_f = & \sum_{\nu} \sum_V \left[\sum_{R\nu} \psi_{R\nu}^\dagger \psi_{R\nu} + \sum_{Le} \psi_{Le}^\dagger \psi_{Le} \right] \\
 & - m \left(\sum_{Le} \psi_{Le}^\dagger \psi_{e\nu} + \sum_{R\nu} \psi_{R\nu}^\dagger \psi_{e\nu} + \text{h.c.} \right)
 \end{aligned}$$

運動方程式

$$\not{D}_{Rv}^2 + m^2 (E E^T)_{Rv} = 0$$

$$\not{D}_{Le}^2 + m^2 (E^T E)_{Le} = 0$$

right-handed Weyl フェルミオン	1個
left-handed Weyl フェルミオン	$q - p + 1$ 個
massive Dirac フェルミオン	$p - 1$ 個

局所対称性 + U(1)リンク場との結合

ゲージ変換 $R_v W_v R_v^{-1}$, $Le W_{o(e)} Le^{-1}$ ($W_v = e^{i v}$)

$$\begin{aligned}
 L_f &= \int_{v V} \bar{\psi}_{Rv} i \not{D}_\mu Rv \psi_{Rv} + \int_{e E} \bar{\psi}_{Le} i \not{D}_\mu Le \psi_{Le} \\
 &\quad - m \left[\int_{e E} \bar{\psi}_{Le} (R_{o(e)} - U_e R_{t(e)}) \psi_{Rv} + h.c. \right] \\
 &= \int_{v V} \bar{\psi}_{Rv} i \not{D}_\mu Rv \psi_{Rv} + \int_{e E} \bar{\psi}_{Le} i \not{D}_\mu Le \psi_{Le} \\
 &\quad - m \left(\int_{e E} \bar{\psi}_{Le} \hat{E}_{ev}^\dagger \psi_{Rv} + h.c. \right)
 \end{aligned}$$

ここで

$$(\hat{E})_{ve} = \begin{cases} 1 & \text{if } v = o(e) \\ -U_e^{\dagger} & \text{if } v = t(e) \\ 0 & \text{otherwise} \end{cases}$$

modified graph Laplacian

$$(\hat{L})_{vv'} = \begin{cases} \text{deg}(v) & \text{if } v = v' \\ -U_e^{\dagger} & \text{if } e = [v', v] \text{ exists} \\ -U_e & \text{if } e = [v, v'] \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

low-energy ($E \ll f, m$)

$$L_{\text{int}} = \sum_v \frac{1}{V} \sum_{\mu} A_{\nu}^{\mu} + \sum_e \frac{1}{E} \sum_{\mu} A_{o(e)}^{\mu} - \frac{m}{f} \left(\sum_e \sum_{\mu} X_{e, R t(e)} + \text{h.c.} \right)$$

$p = q$ の場合 閉路 $\tilde{C}_{\tilde{p}}$ ($\tilde{p} = p$) を必ず 1 つだけ含む
zero-mode fields

$$A_{\nu}^{\mu} = \frac{1}{\sqrt{p}} A_{o}^{\mu}, \quad R_{\nu} = \frac{1}{\sqrt{p}} R_0 \quad \text{for all } \nu$$

$$X_{e} = \frac{1}{\sqrt{\tilde{p}}} X_{o}, \quad L_e = \frac{1}{\sqrt{\tilde{p}}} L_0 \quad \text{for all } e \in E(\tilde{C}_{\tilde{p}})$$

zero-mode interactions

$$\begin{aligned}
 L_{\text{int}0} = & \frac{1}{\sqrt{p}} \bar{\psi}_{R0} \gamma^\mu A_0^\mu \psi_{R0} + \frac{1}{\sqrt{p}} \bar{\psi}_{L0} \gamma^\mu A_0^\mu \psi_{L0} \\
 & - \frac{1}{\sqrt{p}} \frac{m}{f} (\bar{\psi}_{L0} X_0 \psi_{R0} + \text{h.c.})
 \end{aligned}$$

the gauge coupling = $\frac{g}{\sqrt{p}}$, the link scalar coupling = $\frac{1}{\sqrt{p}} \frac{m}{f}$

$m = gf$ で同じオーダー

§ 5. ONE-LOOP DIVERGENCES IN EFFECTIVE ACTION

effective actionをheat-kernelの方法で計算して
one-loopの発散を調べる。(metricはEuclidean)

$$= \begin{bmatrix} & R \\ & \\ L & \end{bmatrix}, \quad i\mathbb{D} = \begin{bmatrix} iD_v & -m\hat{E} \\ -m\hat{E}^\dagger & iD_e \end{bmatrix}$$

ここで

$$D_v = \text{diag} \left(\mu D_{v_1}^\mu, \mu D_{v_2}^\mu, \dots, \mu D_{v_p}^\mu \right)$$

$$D_e = \text{diag} \left(\mu D_{e_1}^\mu, \mu D_{e_2}^\mu, \dots, \mu D_{e_q}^\mu \right)$$

$$L_f = \overline{i\mathbb{D}}$$

$$D^\dagger D = \begin{bmatrix} D_V^2 + m^2 \hat{E} \hat{E}^\dagger & \text{im}(D_V E - E D_E) \\ \text{im}(D_E \hat{E}^\dagger - \hat{E}^\dagger D_V) & D_E^2 + m^2 \hat{E}^\dagger \hat{E} \end{bmatrix}$$

ここで

$$(D_V^2)_{vv'} = \begin{cases} - (D_v^\mu)^2 - \frac{i}{2} \mu F_v^\mu & \text{if } v = v' \\ 0 & \text{otherwise} \end{cases}$$

$$(D_E^2)_{ee'} = \begin{cases} - (D_e^\mu)^2 - \frac{i}{2} \mu F_{\alpha(e)}^\mu & \text{if } e = e' \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 (D_v E - E D_E)_{ve} &= \begin{cases} -\mu (U_e)^\dagger & \text{if } v = t(e) \\ 0 & \text{otherwise} \end{cases} \\
 (D_E \hat{E}^\dagger - \hat{E}^\dagger D_V)_{ev} &= \begin{cases} -\mu (U_e) & \text{if } v = t(e) \\ 0 & \text{otherwise} \end{cases}
 \end{aligned}$$

effective action at one-loop level

$$= -\frac{1}{2} \text{Tr} \ln (\mathbb{D}^\dagger \mathbb{D}) = \frac{1}{2} \int_0^{\infty} \frac{dt}{t} \text{Tr} e^{-\mathbb{D}^\dagger \mathbb{D} t}$$

t^2 に比例するところ

$$\text{tr} (D_v^2)^2 = 2 \int_V \left[(D_v^\mu)^2 (D_v)^2 + \frac{1}{2} F_v^\mu F_v^\mu \right]$$

$$\text{tr} (D_E^2)^2 = 2 \int_E \left[(D_e^\mu)^2 (D_e)^2 + \frac{1}{2} F_{o(e)}^\mu F_{o(e)}^\mu \right]$$

logarithmic divergent part

$$\frac{1}{24} \frac{dt}{t} \left[\sum_{\nu} \frac{1}{4} (F_{\nu}^{\mu})^2 + \frac{1}{4} (F_{\alpha(e)}^{\mu})^2 + \frac{3}{2} m_e^2 (U_e)^2 + \dots \right]$$

gauge coupling runs logarithmically

$d^+(v) = 1$ for all v のとき $\mu \frac{dg}{d\mu} = \frac{g^3}{12}$ QEDと同じ

各々頂点で個々のgauge couplingを定義すると $\mu \frac{dg_{\nu}}{d\mu} = \frac{(1 + n_{\nu}) g_{\nu}^3}{24}$

§ 6. ONE-LOOP FINITENESS IN EFFECTIVE POTENTIAL

The effective potential for constant background link field

$$\begin{aligned}
 V &= - \int_0^{\infty} \frac{dt}{t} \int \frac{d^4 k}{(2\pi)^4} e^{-k^2 t} \left[\text{tr}_p \exp(-m^2 \hat{E} \hat{E}^\dagger t) + \text{tr}_q \exp(-m^2 \hat{E}^\dagger \hat{E} t) \right] \\
 &= - \frac{1}{(4\pi)^2} \int_0^{\infty} \frac{dt}{t^3} \left[\text{tr}_p \exp(-m^2 \hat{E} \hat{E}^\dagger t) + \text{tr}_q \exp(-m^2 \hat{E}^\dagger \hat{E} t) \right]
 \end{aligned}$$

$$\text{tr}_p (\hat{E} \hat{E}^\dagger)^n = \text{tr}_q (\hat{E}^\dagger \hat{E})^n \quad (n \geq 1) \text{ より}$$

$$\begin{aligned} & \frac{1}{2} \left[\text{tr}_p \exp(-m^2 \hat{E} \hat{E}^\dagger t) + \text{tr}_q \exp(-m^2 \hat{E}^\dagger \hat{E} t) \right] \\ &= \frac{p+q}{2} - m^2 \text{tr}_p \hat{\Delta} t + \frac{1}{2} m^4 \text{tr}_p \hat{\Delta}^2 t^2 + O(t^3) \end{aligned}$$

ここで

$$\text{tr}_p \hat{\Delta} = \text{tr}_p D, \quad \text{tr}_p \hat{\Delta}^2 = \text{tr}_p D^2 + \text{tr}_p D$$

発散はlink fieldsに依らない

スカラーポテンシャルはone-loopで有限

§ 7. SUMMARY & PROSPECTS

Graph上の場の理論の枠組みで
one-loop effective lagrangianの発散を
明らかにしました。

- (i) 基本タイセット行列

- non-Abelianへの拡張
- adjoint matter field
- plaquette-like term
- two-loop

etc.

Spectral graph theoryの知識 高次のloopの発散を
調べるのに有益