Weyl変換，Weyl tensor

1999年07月07日改訂

概 要

1 曲率の定義

Riemann tensor:

$$R^{\mu}{}_{\sigma\beta\alpha} = \partial_{\beta}\Gamma^{\mu}_{\sigma\alpha} - \partial_{\alpha}\Gamma^{\mu}_{\sigma\beta} + \Gamma^{\mu}_{\rho\beta}\Gamma^{\rho}_{\sigma\alpha} - \Gamma^{\mu}_{\rho\alpha}\Gamma^{\rho}_{\sigma\beta} \tag{1}$$

ここで

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\nu\mu}\right) \tag{2}$$

Ricci tensor:

$$R_{\sigma\alpha} \equiv R^{\mu}{}_{\sigma\mu\alpha} \tag{3}$$

scalar curvature:

$$R \equiv g^{\sigma\alpha}R_{\sigma\alpha} \tag{4}$$

2 Weyl変換

Weyl 変換:

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu} \tag{5}$$

このとき

$$
\begin{aligned}
\bar{R}^{\mu}{}_{\sigma\beta\alpha} &= R^{\mu}{}_{\sigma\beta\alpha} + \delta^{\mu}_{\alpha}\nabla_{\beta}\nabla_{\sigma}\omega - \delta^{\mu}_{\beta}\nabla_{\alpha}\nabla_{\sigma}\omega \\
&- g_{\sigma\alpha}\nabla_{\beta}\nabla^{\mu}\omega + g_{\sigma\beta}\nabla_{\alpha}\nabla^{\mu}\omega - \delta^{\mu}_{\alpha}\nabla_{\beta}\omega\nabla_{\sigma}\omega \\
&+ \delta^{\mu}_{\beta}\nabla_{\alpha}\omega\nabla_{\sigma}\omega + g_{\sigma\alpha}\nabla_{\beta}\omega\nabla^{\mu}\omega - g_{\sigma\beta}\nabla_{\alpha}\omega\nabla^{\mu}\omega \\
&- \delta^{\mu}_{\beta}g_{\sigma\alpha}\nabla_{\lambda}\omega\nabla^{\lambda}\omega + \delta^{\mu}_{\alpha}g_{\sigma\beta}\nabla_{\lambda}\omega\nabla^{\lambda}\omega
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\bar{R}_{\mu\nu} &= R_{\mu\nu} - (D-2)\nabla_{\mu}\nabla_{\nu}\omega - g_{\mu\nu}\nabla^{\lambda}\nabla_{\lambda}\omega \\
&+ (D-2)\nabla_{\mu}\omega\nabla_{\nu}\omega - (D-2)g_{\mu\nu}\nabla_{\lambda}\omega\nabla^{\lambda}\omega
\end{aligned} \tag{7}
$$

$$
\bar{R} = e^{-2\omega}\left[R - 2(D-1)\nabla^{\lambda}\nabla_{\lambda}\omega - (D-1)(D-2)\nabla_{\lambda}\omega\nabla^{\lambda}\omega\right] \tag{8}
$$

<u>exercise</u>

scalar field ϕ が Weyl 変換の下で

$$
\phi \to \bar{\phi} = e^{-\frac{D-2}{2}\omega}\phi \tag{9}
$$

と変換するものとする。

このとき

$$
-\frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}\phi\right) + \frac{1}{4}\frac{D-2}{D-1}R\phi = 0 \tag{10}
$$

は，Weyl「不変」な方程式であることを示せ。

<u>exercise</u>

action

$$
S = \int d^D x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{8}\frac{D-2}{D-1}R\phi^2\right] \tag{11}
$$

は，Weyl 不変であることを示せ。

また，Weyl 不変な ϕ の自己相互作用項をつくれ。

3 Weyl tensor

Weyl tensor:

$$\begin{aligned} C_{\mu\sigma\beta\alpha} &= R_{\mu\sigma\beta\alpha} \\ &+ \frac{1}{D-2}\left(g_{\mu\alpha}R_{\sigma\beta} + g_{\sigma\beta}R_{\mu\alpha} - g_{\mu\beta}R_{\sigma\alpha} - g_{\sigma\alpha}R_{\mu\beta}\right) \\ &+ \frac{1}{(D-1)(D-2)}\left(g_{\mu\beta}g_{\sigma\alpha} - g_{\mu\alpha}g_{\sigma\beta}\right)R \end{aligned} \tag{12}$$

$$\bar{C}_{\mu\sigma\beta\alpha} = e^{2\omega}C_{\mu\sigma\beta\alpha} \tag{13}$$

対称性:

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta} \tag{14}$$

$$C_{\alpha\beta\gamma\delta} = -C_{\beta\alpha\gamma\delta} = -C_{\alpha\beta\delta\gamma} \tag{15}$$

$$C_{\alpha\beta\gamma\delta} + C_{\alpha\delta\beta\gamma} + C_{\alpha\gamma\delta\beta} = 0 \tag{16}$$

$$C^{\alpha}{}_{\beta\alpha\delta} \equiv 0 \tag{17}$$

Weyl tensor の独立な成分の個数は，Riemann tensor の独立成分の個数から (17) の制限の個数をひいたものであるので，

$$\frac{D^2(D^2-1)}{12} - \frac{D(D+1)}{2} = \frac{D(D+1)(D+2)(D-3)}{12} \tag{18}$$

である。($D \geq 3$)

<u>exercise</u>

３次元では

$$C_{\alpha\beta\gamma\delta} \equiv 0 \tag{19}$$

となることを示せ。