

# Graph Laplacian and Graph Hosotani Mechanism?<sup>1</sup>

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Graph Laplacian Let  $G(V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . The graph Laplacian (or combinatorial Laplacian)  $\Delta$  is defined [1] by

$\Delta_{uv} = (D - A)_{uv} = \text{deg}(v)$  if  $u = v$ ;  $-1$  if  $u$  and  $v$  are adjacent ( $u \sim v$ );  $0$  otherwise, where  $u, v \in V$  and  $\text{deg}(v)$  denotes the degree of  $v$ . The diagonal matrix  $D$  is called as the degree matrix while  $A$  is known as the adjacency matrix.

Field Theory on a Graph (Graph Field Theory=GFT) We can construct the action for various free fields whose (mass)<sup>2</sup> spectra are given by the eigenvalues of  $\Delta$  (an associated mass scale is set to unity) [2]. For example, the lagrangian density for scalar fields can be written by

$$\mathcal{L}_s = -\frac{1}{2} \sum_{u,v} \phi_u(x) (-\nabla^2 \delta_{uv} + \Delta_{uv}) \phi_v(x) = -\frac{1}{2} \sum_v (\nabla_\mu \phi_v(x))^2 - \frac{1}{4} \sum_{u \sim v} (\phi_u(x) - \phi_v(x))^2.$$

Next we consider a directed graph. An oriented edge  $e = \{u, v\}$  connects the origin  $u = o(e)$  and the terminus  $v = t(e)$ . The lagrangian density for Dirac fields associated with the directed graph can be written by

$$\mathcal{L}_f = \sum_{v \in V} \bar{\psi}_{Rv} i \gamma^\mu \partial_\mu \psi_{Rv} + \sum_{e \in E} \bar{\psi}_{Le} i \gamma^\mu \partial_\mu \psi_{Le} + (\sum_{e \in E} \sum_{v \in V} \bar{\psi}_{Le} E_{ev}^\dagger \psi_{Rv} + h.c.),$$

where the incidence matrix  $E$  is defined by  $E_{ve} = 1$  if  $v = o(e)$ ;  $-1$  if  $v = t(e)$ ;  $0$  otherwise. Since  $EE^\dagger = \Delta$ , the mass spectrum is given by the eigenvalues of  $\Delta$  and  $(\#E - \#V)$  zero modes. If  $\#E = \#V$ , the left-handed fermion may be put on each vertex. In this case, the mass term becomes  $(\sum_{e \in E} \bar{\psi}_{L o(e)} (\psi_{R o(e)} - \psi_{R t(e)}) + h.c.)$ . This is the same as the term for the Wilson fermion with a unit hopping parameter if we take a cycle graph ( $C_n$ ).

Further we introduce link fields  $U_e$  (with  $|U_e| = 1$ ). We can write the lagrangian density for vector fields whose (mass)<sup>2</sup> matrix is  $\Delta$  (and, a certain number of massless scalar fields) as

$$\mathcal{L}_v = -\frac{1}{2} \sum_{v \in V} \text{tr} F_v^{\mu\nu} F_{v\mu\nu} - \sum_{e \in E} \text{tr} |D^\mu U_e|^2, \text{ where } D^\mu U_e = \partial^\mu U_e + i(A_{o(e)}^\mu U_e - U_e A_{t(e)}^\mu).$$

Introducing the link field into matter-field lagrangian is also possible. Then  $\Delta_{o(e)t(e)}$  is replaced by  $-U_e$ , instead of  $-1$  (in other words, the edge  $e$  has the edge-weight  $U_e$ ).

One-loop divergences in effective action One-loop effective action can be obtained by use of the heat kernel method. Because the trace of the kernel for a graph Laplacian can be expanded as

$$\text{Tr} \exp(-\Delta t) = \#V - (\text{Tr} D)t + \frac{1}{2}(\text{Tr} D^2 + \text{Tr} D)t^2 + O(t^3),$$

the amount of quartic, quadratic and logarithmic divergences in the effective action for GFT should depend only on the number of vertices (the order of  $G$ ) and the degree matrix (up to zero-mode contribution). One can choose matter contents and graph for each field in order to cancel the divergences; the divergences from bosonic and fermionic degrees of freedom on graphs with the same degree matrix cancel each other. An application to one-loop finite ‘induced gravity’ can be found in [3].

Graph Hosotani Mechanism? Let the length of the shortest closed path  $c = (e_1, \dots, e_N)$  in  $G$  be  $N(\geq 3)$ . The trace of the heat kernel of  $\Delta$  for matter fields coupled to the link fields includes a term  $\text{ReTr} U_{e_1} \cdots U_{e_N}$  and its coefficient is  $O(t^N)$ . Therefore the one-loop effective potential for the zero mode of link fields is UV finite (up to a field-independent divergence). If we take  $C_n$  and consider the limit  $n \rightarrow \infty$ , the model reduces to the original Hosotani model [4] (it should be read  $A_5 \sim \chi$ , where  $U = e^{i\chi}$ ). The realization of non-Abelian symmetry breaking in GFT may not be so easy as in the Hosotani models. We are studying the symmetry breaking in GFT of the graph which has edges with and without weights (a simple example has been used in [3]). The report on the symmetry breaking in GFT will appear elsewhere [5].

[1] B. Mohar, in *Graph Theory, Combinatorics, and Applications*, Y. Alavi et al. (eds.), Wiley, New York, pp.871-898 (1991); in *Graph Symmetry, Algebraic Methods, and Applications*, G. Hahn and G. Sabidussi (eds.), Kluwer, Dordrecht, pp.225-275 (1997).

[2] For discussion on the mass spectrum, see N. Kan, in this proceedings.

[3] N. Kan and K. Shiraishi, ‘Induced Gravity from Theory Space’, gr-qc/0310055.

[4] Y. Hosotani, Phys. Lett. **B126** (1983) 309.

[5] N. Kan and K. Shiraishi, in preparation.

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