

曲率の計算：球対称な場合

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概要

Einstein 方程式の球対称な解を求めるために。

1 時空

1.1 $(N + 1)$ 次元時空の metric

球対称

$$ds^2 = -e^{-2\delta} \Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_{N-1}^2 \quad (1)$$

Δ, δ は r のみの関数と仮定する。

1.2 vielbein

$$e^0 = e^{-\delta} \sqrt{\Delta} dt \quad (2)$$

$$e^1 = \frac{1}{\sqrt{\Delta}} dr \quad (3)$$

$$e^A = r \tilde{e}^A \quad (4)$$

ここで

$$d\tilde{e}^A + \tilde{\omega}^A{}_B \wedge \tilde{e}^B = 0 \quad (5)$$

$A, B = 2, \dots, N$ とする。

$$de^0 = \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) dr \wedge e^0 = \sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) e^1 \wedge e^0 \quad (6)$$

$$de^1 = 0 \quad (7)$$

$$de^A = \frac{1}{r} dr \wedge e^A + r d\tilde{e}^A = \frac{\sqrt{\Delta}}{r} e^1 \wedge e^A - \tilde{\omega}^A{}_B \wedge e^B \quad (8)$$

1.3 spin connection

$$\omega^0_{\ 1} = \sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) e^0 \quad (9)$$

$$\omega^A_{\ 1} = \frac{\sqrt{\Delta}}{r} e^A \quad (10)$$

$$\omega^A_{\ B} = \tilde{\omega}^A_{\ B} \quad (11)$$

1.4 curvature 2-form

$$\begin{aligned} \Theta^0_{\ 1} &= d\omega^0_{\ 1} \\ &= - \left\{ \sqrt{\Delta} \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]' + \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]^2 \right\} e^0 \wedge e^1 \end{aligned} \quad (12)$$

$$\begin{aligned} \Theta^A_{\ 1} &= d\omega^A_{\ 1} + \omega^A_{\ B} \wedge \omega^B_{\ 1} \\ &= - \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} e^A \wedge e^1 \end{aligned} \quad (13)$$

$$\begin{aligned} \Theta^A_{\ B} &= d\omega^A_{\ B} + \omega^A_{\ C} \wedge \omega^C_{\ B} + \omega^A_{\ 1} \wedge \omega^1_{\ B} \\ &= \left\{ \frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} e^A \wedge e^B \end{aligned} \quad (14)$$

$$\begin{aligned} \Theta^0_{\ A} &= \omega^0_{\ 1} \wedge \omega^1_{\ A} \\ &= - \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right] \left[\frac{\sqrt{\Delta}}{r} \right] e^0 \wedge e^A \end{aligned} \quad (15)$$

1.5 Riemann tensor

$$R^{01}{}_{01} = - \left\{ \sqrt{\Delta} \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]' + \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]^2 \right\} \quad (16)$$

$$R^{A1}{}_{B1} = - \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \delta_B^A \quad (17)$$

$$R^{AB}{}_{CD} = \left\{ \frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} (\delta_C^A \delta_D^B - \delta_D^A \delta_C^B) \quad (18)$$

$$R^{0A}{}_{0B} = - \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right] \left[\frac{\sqrt{\Delta}}{r} \right] \delta_B^A \quad (19)$$

1.6 Ricci tensor

$$\begin{aligned}
R_0^0 &= R^{01}_{01} + R^{0A}_{0A} \\
&= - \left\{ \sqrt{\Delta} \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]' + \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]^2 \right\} \\
&\quad - (N-1) \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right] \left[\frac{\sqrt{\Delta}}{r} \right]
\end{aligned} \tag{20}$$

$$\begin{aligned}
R_1^1 &= R^{01}_{01} + R^{A1}_{A1} \\
&= - \left\{ \sqrt{\Delta} \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]' + \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]^2 \right\} \\
&\quad - (N-1) \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\}
\end{aligned} \tag{21}$$

$$\begin{aligned}
R_B^A &= R^{0A}_{0B} + R^{A1}_{B1} + R^{AC}_{BC} \\
&= \left(- \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right] \left[\frac{\sqrt{\Delta}}{r} \right] \right. \\
&\quad \left. - \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \right) \\
&\quad + (N-2) \left\{ \frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \delta_B^A
\end{aligned} \tag{22}$$

1.7 scalar curvature

$$\begin{aligned}
R &= -2 \left\{ \sqrt{\Delta} \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]' + \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]^2 \right\} \\
&\quad - 2(N-1) \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right] \left[\frac{\sqrt{\Delta}}{r} \right] \\
&\quad - 2(N-1) \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \\
&\quad + (N-1)(N-2) \left\{ \frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \tag{23}
\end{aligned}$$

1.8 Einstein tensor

$$G_\nu^\mu \equiv R_\nu^\mu - \frac{1}{2} R \delta_\nu^\mu \tag{24}$$

$$\begin{aligned}
G_0^0 &= (N-1) \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \\
&\quad - \frac{(N-1)(N-2)}{2} \left\{ \frac{1}{r^2} - \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \tag{25}
\end{aligned}$$

$$\begin{aligned}
G_0^0 - G_1^1 &= R_0^0 - R_1^1 \\
&= -(N-1) \left[\sqrt{\Delta} \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right] \left[\frac{\sqrt{\Delta}}{r} \right] \\
&\quad + (N-1) \left\{ \sqrt{\Delta} \left[\frac{\sqrt{\Delta}}{r} \right]' + \left[\frac{\sqrt{\Delta}}{r} \right]^2 \right\} \tag{26}
\end{aligned}$$

1.9 使う式

$$G_0^0 = \frac{N-1}{2r} \Delta' + \frac{(N-1)(N-2)}{2} \frac{\Delta-1}{r^2} \quad (27)$$

$$G_0^0 - G_1^1 = \Delta \frac{N-1}{r} \delta' \quad (28)$$

2 Einstein equation

$$G_{\nu}^{\mu} = R_{\nu}^{\mu} - \frac{1}{2}R\delta_{\nu}^{\mu} = 8\pi GT_{\nu}^{\mu} \quad (29)$$

($N + 1$) 次元では

$$R_{\nu}^{\mu} = 8\pi G \left(T_{\nu}^{\mu} - \frac{1}{N-1} T\delta_{\nu}^{\mu} \right) \quad (30)$$

完全流体 (perfect fluid) を仮定すると

$$T_{\nu}^{\mu} = \text{diag.} (-\rho, P, P, \dots, P) \quad (31)$$

とすると

$$T = T_{\mu}^{\mu} = -\rho + NP \quad (32)$$

$$G_0^0 = -8\pi G\rho \quad (33)$$

$$G_0^0 - G_1^1 = -8\pi G(\rho + P) \quad (34)$$

$$(35)$$

$$\frac{N-1}{2r}\Delta' + \frac{(N-1)(N-2)}{2} \frac{\Delta-1}{r^2} = -8\pi G\rho \quad (36)$$

$$\Delta \left[\frac{N-1}{r} \delta' \right] = -8\pi G(\rho + P) \quad (37)$$

3 conservation

$$\nabla_\lambda T^{\mu\nu} = \partial_\lambda T^{\mu\nu} + \Gamma_{\lambda\sigma}^\mu T^{\sigma\nu} + \Gamma_{\lambda\sigma}^\nu T^{\mu\sigma} \quad (38)$$

これより

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \partial_\mu T^{\mu\nu} + \Gamma_{\mu\sigma}^\mu T^{\sigma\nu} + \Gamma_{\mu\sigma}^\nu T^{\mu\sigma} \\ &= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} T^{\mu\nu}) + \Gamma_{\mu\sigma}^\nu T^{\mu\sigma} \end{aligned} \quad (39)$$

ここで

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} = \frac{1}{2} g^{\lambda\sigma} \partial_\mu g_{\lambda\sigma} = \Gamma_{\mu\lambda}^\lambda \quad (40)$$

を使った。

$$T^{tt} = \frac{1}{e^{-2\delta} \Delta} \rho \quad (41)$$

$$T^{rr} = \Delta P \quad (42)$$

$$T^{ij} = \frac{1}{r^2} \tilde{g}^{ij} P \quad (43)$$

$$(44)$$

および

$$\Gamma_{tt}^r = e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \quad (45)$$

$$\Gamma_{rr}^r = -\frac{1}{2} \frac{\Delta'}{\Delta} \quad (46)$$

$$\Gamma_{ij}^r = -\Delta r \tilde{g}_{ij} \quad (47)$$

$$(48)$$

などを使うと、保存の式、あるいは力学的平衡の式

$$\nabla_\mu T^{\mu r} = 0 \quad (49)$$

は次のようになる。

$$P' + \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) (\rho + P) = 0 \quad (50)$$

4 重力平衡の式

アインシュタイン方程式とあわせると,

$$-P' = \frac{8\pi G}{(N-1)\Delta} r \left(\frac{(N-1)(N-2)}{16\pi G} \frac{1-\Delta}{r^2} + P \right) (\rho + P) \quad (51)$$

$$\Delta = 1 - \frac{16\pi G M_r}{(N-1)A_{N-1}r^{N-2}} \quad (52)$$

とおくと

$$-P' = \frac{8\pi G}{(N-1) \left(1 - \frac{16\pi G M_r}{(N-1)A_{N-1}r^{N-2}}\right)} r \left(\frac{(N-2)M_r}{A_{N-1}r^N} + P \right) (\rho + P) \quad (53)$$

ニュートン近似では

$$-P' = \frac{8\pi G(N-2)M_r}{(N-1)A_{N-1}r^{N-1}} \rho \quad (54)$$

を得る。

なお,

$$M_r(r) = A_{N-1} \int_0^r \rho(r') r'^{N-1} dr' \quad (55)$$

である。

$$A_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} \quad (56)$$

Appendix

$$ds^2 = -e^{-2\delta} \Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_{N-1}^2 \quad (57)$$

Δ, δ は r のみの関数と仮定する。

$$d\Omega_{N-1}^2 = \tilde{g}_{ij} d\tilde{x}^i d\tilde{x}^j \quad (i, j = 2, \dots, N) \quad (58)$$

$$\Gamma_{rt}^t = -\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \quad (59)$$

$$\Gamma_{tt}^r = e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \quad (60)$$

$$\Gamma_{rr}^r = -\frac{1}{2} \frac{\Delta'}{\Delta} \quad (61)$$

$$\Gamma_{ij}^r = -\Delta r \tilde{g}_{ij} \quad (62)$$

$$\Gamma_{rj}^i = \frac{1}{r} \delta_j^i \quad (63)$$

$$\Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i \quad (64)$$

$$\Gamma_{t\lambda}^\lambda = 0 \quad (65)$$

$$\Gamma_{r\lambda}^\lambda = \Gamma_{rt}^t + \Gamma_{rr}^r + \Gamma_{rk}^k = -\delta' + \frac{N-1}{r} \quad (66)$$

$$\Gamma_{i\lambda}^\lambda = \tilde{\Gamma}_{ik}^k \quad (67)$$

$$R_{\sigma\nu} = \partial_\lambda \Gamma_{\nu\sigma}^\lambda - \partial_\nu \Gamma_{\sigma\lambda}^\lambda + \Gamma_{\rho\lambda}^\lambda \Gamma_{\nu\sigma}^\rho - \Gamma_{\nu\rho}^\lambda \Gamma_{\sigma\lambda}^\rho \quad (68)$$

$$\begin{aligned} R_{tt} &= \partial_r \Gamma_{tt}^r - 0 + \Gamma_{tt}^r \Gamma_{r\lambda}^\lambda - 2\Gamma_{tt}^r \Gamma_{tr}^t \\ &= \left[e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \right]' \end{aligned}$$

$$\begin{aligned}
& + e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \left[-\delta' + \frac{N-1}{r} \right] \\
& - 2e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right)^2 \\
& = e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right)' \\
& + e^{-2\delta} \Delta^2 \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right) \left[-\delta' + \frac{N-1}{r} + \frac{\Delta'}{\Delta} \right] \quad (69)
\end{aligned}$$

$$\begin{aligned}
R_{rr} & = \partial_r \Gamma_{rr}^r - \partial_r \Gamma_{r\lambda}^\lambda + \Gamma_{rr}^r \Gamma_{r\lambda}^\lambda - \left(\Gamma_{rt}^t{}^2 + \Gamma_{rr}^r{}^2 + \Gamma_{rj}^i \Gamma_{ri}^j \right) \\
& = \left(-\frac{1}{2} \frac{\Delta'}{\Delta} \right)' - \left[-\delta' + \frac{N-1}{r} \right]' + \left(-\frac{1}{2} \frac{\Delta'}{\Delta} \right) \left[-\delta' + \frac{N-1}{r} \right] \\
& - \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right)^2 - \left(-\frac{1}{2} \frac{\Delta'}{\Delta} \right)^2 - \frac{N-1}{r^2} \\
& = - \left[-\delta' + \frac{N-1}{r} + \frac{1}{2} \frac{\Delta'}{\Delta} \right]' + \left(-\frac{1}{2} \frac{\Delta'}{\Delta} \right) \left[-\delta' + \frac{N-1}{r} + \frac{1}{2} \frac{\Delta'}{\Delta} \right] \\
& - \left(-\delta' + \frac{1}{2} \frac{\Delta'}{\Delta} \right)^2 - \frac{N-1}{r^2} \quad (70)
\end{aligned}$$

$$\begin{aligned}
R_{ij} & = \partial_r \Gamma_{ij}^r + \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{j\lambda}^\lambda + \Gamma_{ij}^r \Gamma_{r\lambda}^\lambda + \Gamma_{ij}^k \Gamma_{k\lambda}^\lambda - \left(2\Gamma_{ri}^k \Gamma_{jk}^r + \Gamma_{li}^k \Gamma_{jk}^\ell \right) \\
& = [-\Delta r]' \tilde{g}_{ij} + \partial_k \tilde{\Gamma}_{ij}^k - \partial_i \tilde{\Gamma}_{jk}^k + [-\Delta r] \left[-\delta' + \frac{N-1}{r} \right] \tilde{g}_{ij} \\
& + \tilde{\Gamma}_{ij}^k \tilde{\Gamma}_{k\ell}^\ell - 2[-\Delta] \tilde{g}_{ij} - \tilde{\Gamma}_{li}^k \tilde{\Gamma}_{jk}^\ell \\
& = \tilde{R}_{ij} - \Delta r \left[-\delta' + \frac{N-2}{r} + \frac{\Delta'}{\Delta} \right] \tilde{g}_{ij} \quad (71)
\end{aligned}$$

参考文献

[1] ???