# Scattering of maximally-charged dilatonic black holes

Kiyoshi Shiraishi (Yamaguchi Univ.)

**TOC**

# Introduction

**Solving the many-body (even for 2-body) problem of BHs in General Relativity is very difficult.**

**Maximally Charged BHs can move slowly even in the strong gravitational field,**

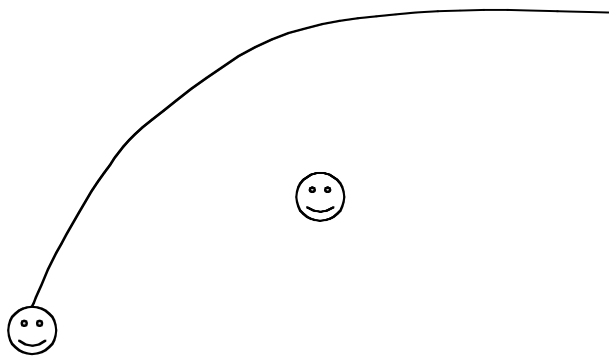
**because the cancellation of static forces.**

**Their motion can be described as geodesics in the moduli space.**

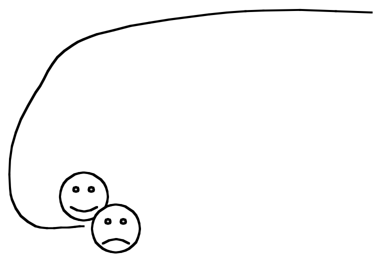
**On the other hand, dilaton-coupled BPS solitons often appear in String Theory.**

We analyze two-body scattering of maximally-charged dilatonic black holes by studying their moduli space.

Scattering off to the infinity



Coalescence of two BHs



# 1. Charged Dilatonic Black Holes

The Action for the Einstein-Maxwell-Dilaton system is

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi}\left[R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2\right]$$

a=1: the action coincides with that of effective field theory of string theory.

If we rescale $\quad \bar{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu}$

we get $\quad \bar{S} = \int d^4x \frac{\sqrt{-\bar{g}}}{16\pi} e^{-2\phi}\left[\bar{R} + 4(\bar{\nabla}\phi)^2 - \bar{F}^2\right]$

$a^2=3$: Kaluza-Klein theory for metric in 5 dims.

$$\bar{g}_{AB}dx^A dx^B = \exp\left(\frac{2\phi}{\sqrt{3}}\right) g_{\mu\nu}dx^\mu dx^\nu + \exp\left(\frac{-4\phi}{\sqrt{3}}\right)(dy + 2A_\mu dx^\mu)^2$$

$$\int d^5x \frac{\sqrt{-\bar{g}}}{16\pi L}\bar{R} \rightarrow \int d^4x \frac{\sqrt{-g}}{16\pi}\left[R - 2(\nabla\phi)^2 - e^{-2\sqrt{3}\phi}F^2\right]$$

☆Dilaton fields also appear in certain models of Supergravity.

# The Geometry of a Charged Dilatonic Black Hole

Gibbons and Maeda, Nucl. Phys. B298 (1988) 74l

Garfinkle, Horowitz and Strominger, Phys. Rev. D43 (1991) 3140, D45 (1992) 3888

$$ds^2 = -\Delta\sigma^{-2}dt^2 + \sigma^2(\Delta^{-1}dr^2 + r^2 d\Omega_2^2)$$

where $\Delta(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)$ and $\sigma^2(r) = \left(1 - \frac{r_-}{r}\right)^{2a^2/(1+a^2)}$

Electric Field: $F = \frac{Q}{r^2} dt \wedge dr$

config. of Dilaton Field: $e^{2a\phi} = \sigma^2(r)$

Mass (M): $2M = r_+ + \frac{1-a^2}{1+a^2} r_-$

Electric Charge (Q): $Q^2 = \frac{r_+ r_-}{1+a^2}$

Dilatonic Charge (Σ): $\Sigma = \frac{a r_-}{1+a^2}$

$r_+ = M + \sqrt{M^2 - (1-a^2)Q^2}$ $\quad$ $r_- = \frac{1+a^2}{1-a^2}\left(M - \sqrt{M^2 - (1-a^2)Q^2}\right)$

Extremity Condition:

$$r_+ = r_- \rightarrow Q^2 = (1+a^2)M^2, \quad \Sigma^2 = a^2 M^2$$

$$r_+ = r_- = (1+a^2)M$$

- Area of the horizon:

$$\tilde{A}_H \equiv 4\pi\tilde{r}_+^2$$

with

$$\tilde{r}_+ \equiv r_+\sigma(r_+) = r_+\left(1-\frac{r_-}{r_+}\right)^{a^2/(1+a^2)}$$

- Hawking Temperature:

$$T_H = \beta_H^{-1} = \frac{1}{4\pi r_+}\left(1-\frac{r_-}{r_+}\right)^{(1-a^2)/(1+a^2)}$$

- Entropy of the CDBH:

$$S_0 = \frac{1}{4}\tilde{A}_H = \pi r_+^2\left(1-\frac{r_-}{r_+}\right)^{2a^2/(1+a^2)}$$

Quantum aspects for a charged dilatonic BH:

KS, Mod. Phys. Lett. A7 (1992) 3449

KS, Mod. Phys. Lett. A7 (1992) 3569

KS, Mod. Phys. Lett. A9 (1994) 3509

*************************

There is two pameters for three “charges” (M,Q,Σ).

This is because the condition for existence of a horizon surrounding a singularity.

For general spherically-symmetric soliton solution (with a naked singularity) with arbitrary amounts of charges, see, for example,
M. Rakhmanov, Phys. Rev. D50 (1994) 5155.

*********************

## Rotating Charged Dilaton Black Holes

An exact solution is known only for $a^2=3$ in the above model.
(Belinsky and Ruffini ,Phys. Lettr B89 (1980) 195)

In effective field theory of string theory, exact solutions have been obtained by inclusion of an antisymmetric tensor field.
(A. Sen, Phys. Rev. Lett. 69 (1992) 1006)

In our model with arbitrary a, approximate solutions with infinitesimal spin are known.
(KS, Phys. Lett. A166 (1992) 298,
Horne and Horowitz, Phys. Rev. D46 (1992) 1340)
We find the g factor depends on charge Q if $a\neq 0$ and
g  2 as Q  0 for any a.

Later, solutions with finite angular momentum but small a are found.
(Casadio, Harms, Leblanc and Cox, Phys. Rev. D55 (1997) 814)

# Forces between Charged Particles

on the other hand, Hamiltonian for Two-particle system, obtained by the **Lienard-Wiechert** method

(Landau and Lifschitz, Classical Theory of Fields)

(valid for <u>arbitrary</u> masses, electric charges, and dilatonic charges (KS, J. Math. Phys. 34 (l993) 1480))

$$
\begin{aligned}
H = \frac{1}{2} M_1 \mathbf{v}_1^2 + \frac{1}{2} M_2 \mathbf{v}_2^2 + \frac{1}{R}\left[Q_1 Q_2 - \Sigma_1 \Sigma_2 - M_1 M_2\right] \\
+ \frac{\mathbf{v}_1^2 + \mathbf{v}_2^2}{R}\left[-\Sigma_1 \Sigma_2 + 3 M_1 M_2\right] \\
+ \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{R}\left[Q_1 Q_2 + \Sigma_1 \Sigma_2 - 7 M_1 M_2\right] \\
+ \frac{(\mathbf{n} \cdot \mathbf{v}_1)(\mathbf{n} \cdot \mathbf{v}_2)}{R}\left[Q_1 Q_2 - \Sigma_1 \Sigma_2 - M_1 M_2\right] \\
+ O(\mathbf{v}^3, R^{-2})
\end{aligned}
$$

where R is the distance between two objects <u>1</u> and <u>2</u>.

**For <u>Extreme</u> (Dilatonic) Black Holes ($r_- = r_+$),**

**the mutual *static* forces among BHs are totally cancelled with one another.**

**|Gravitation +Scalar <u>Attractive</u> Force|**

**= |Coulomb repulsion|**

$$M_a M_b + \Sigma_a \Sigma_b = Q_a Q_b = (1 + a^2) M_a M_b$$

$\Downarrow$ suggests...

# Existence of the static multi-BH solution!

cf. the Papapetrou-Majumdar-Myers Solution

(no dilaton)

A. Papapetrou, Proc. A. Irish Acad. A51 (1947) 191.

S. D. Majumdar, Phys. Rev. 72 (1947) 930.

R. C. Myers, Phys. Rev. D35 (1987) 455.

## 2. Static Multi-Centered Solution and Moduli space for Maximally-Charged Dilatonic Black Holes

# Multi-Centered Solution in the isotropic coordinates

KS, J. Math. Phys. 34 (1993) 1480

$$ds^2 = -U^{-2}(\mathbf{x})dt^2 + U^2(\mathbf{x})d\mathbf{x}^2$$

with $U(\mathbf{x}) = [V(\mathbf{x})]^{1/(1+a^2)}$

and $$V(\mathbf{x}) = 1 + \sum_{a=1}^{n} \frac{\mu_a}{|\mathbf{x} - \mathbf{x}_a|}$$

Electric potential : $$A = \frac{1}{\sqrt{1+a^2}}\left\{1 - [V(\mathbf{x})]^{-1}\right\}dt$$

Dilaton configuration: $$e^{-2a\phi} = [V(\mathbf{x})]^{2a^2/(1+a^2)}$$

Mass: $$M_a = \frac{\mu_a}{1+a^2}$$

Electric Charge: $$Q_a = \frac{\mu_a}{\sqrt{1+a^2}}$$

Dilatonic Charge: $$\Sigma_a = \frac{a\mu_a}{1+a^2}$$

**Each "BH" satisfies the Extremity Condition.**

**Cancellation of Static Forces between "BH"s**

**|Gravitation +Scalar <u>Attractive</u> Force|**

**= |Coulomb repulsion|**

$$M_a M_b + \Sigma_a \Sigma_b = Q_a Q_b$$

*************************

Exact solutions for Cosmological Multi-BHs [with Cosmological(-like) term, time-dependent] are also known.

Kastor and Traschen, Phys. Rev. D47 (1993) 5370

Maki and KS, Class. Q. Grav. 10 (1993) 2171

Prog. Theor. Phys. 90 (1993) 1259

Horne and Horowitz, Phys. Rev. D48 (1993) 5457

*************************

# Moduli Space Metric

Static **n**-soliton configuration can be characterized by a finite numbers of parameters.

For Extreme Black Holes, the counting of the number of parameters is easy, since any of two black holes can be distinguished. (cf. Monopoles, Vortices,)

The dimension of the parameter space (moduli space) for n-Extreme BH system is **3n**.

The slow motion of solitons are expected to be described by the **geodesic motion** on the moduli space.

Manton, Phys. Lett. B110 (1982) 54

Ward, Phys. Lett. B158 (1985) 424

Atiyah and Hitchin, “The Geometry and Dynamics of Magnetic Monopoles” (PUP, l988)

To obtain the metric for the moduli space of **n**-maximally charged dilatonic BHs, we calculate the Hamiltonian for the system including the field and **BHs as “sources”**.

Ferrell and Eardley, Phys. Rev. Lett. 59 (1987) 1617

Traschen and Ferrell, Phys. Rev. D45 (1992) 2628

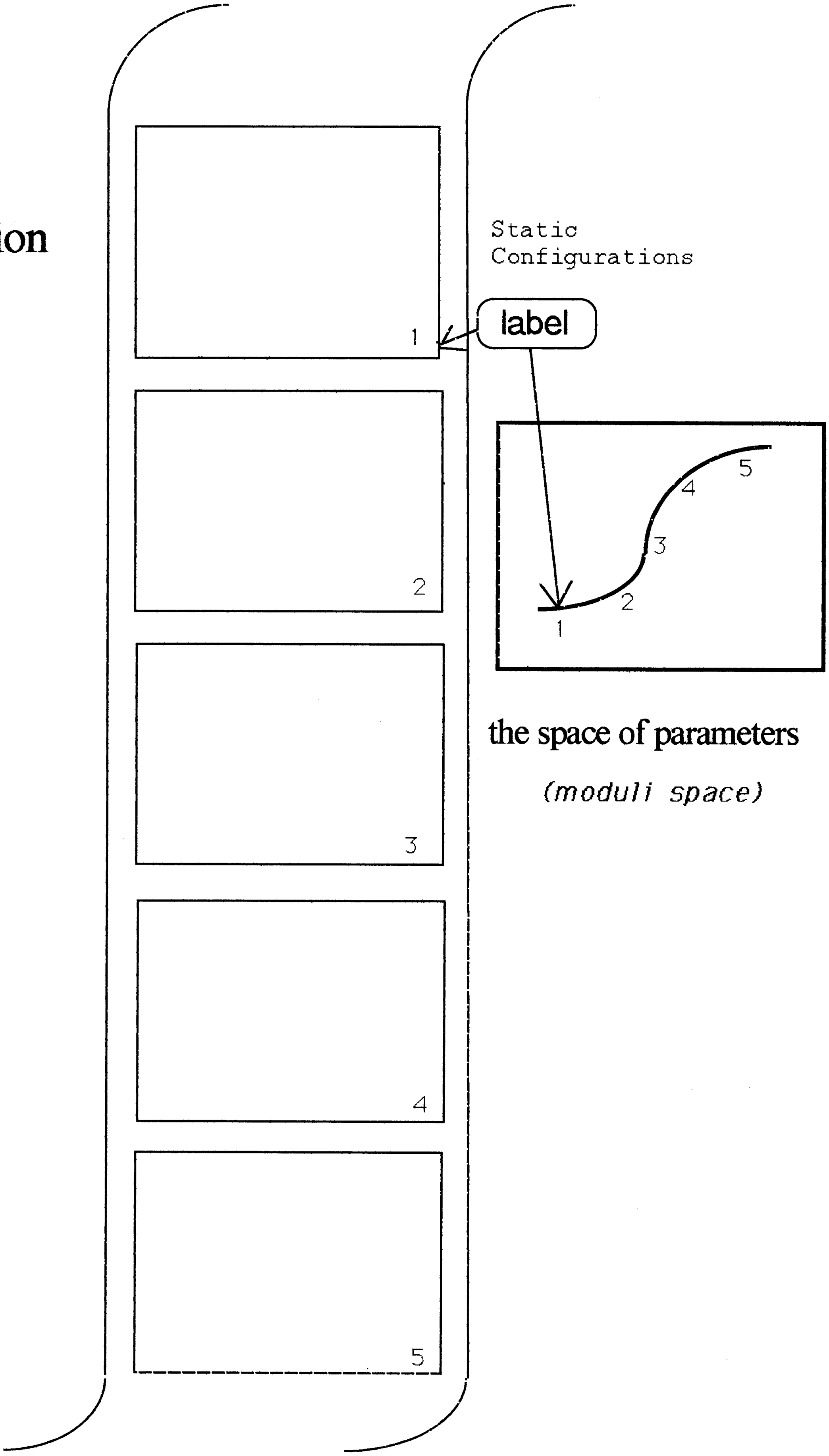
We need only $O(v^2)$ terms. Thus we treat the field equation with slowly-moving "sources" perturbatively. The radiation backreactions are expected to appear in higher order in v.

We can write the $O(v^2)$ Hamiltonian as

$$H = \frac{1}{2}\mathbf{G}_{IJ}\frac{d\mathbf{x}_I}{dt}\frac{d\mathbf{x}_J}{dt}$$

To obtain the trajectory which extremize $\int H dt$ is equivalent to find the geodesic in the space with metric **G**.

# Animation films



*************************

The volume of moduli space is equivalent to Canonical Partition function.

**N. Manton, Nucl. Phys. B400 (1993) 624**

$$Z_N = \frac{1}{N!}\frac{1}{h^{3N}}\int [\mathrm{dq}][\mathrm{dp}]\exp(-\beta H), \qquad H = \frac{1}{2}\mathbf{v}^T\mathbf{G}\mathbf{v} = \frac{1}{2}\mathbf{p}^T\mathbf{G}^{-1}\mathbf{p}$$

$$\Downarrow$$

$$Z_N = \frac{1}{N!}\left(\frac{2\pi}{\beta h^2}\right)^{3N/2}\int [\mathrm{dq}]\sqrt{\det \mathbf{G}}$$

Application to the maximally-charged dilatonic BHs

Kan, Maki and KS, Phys. Rev. D64 (2001) 104009

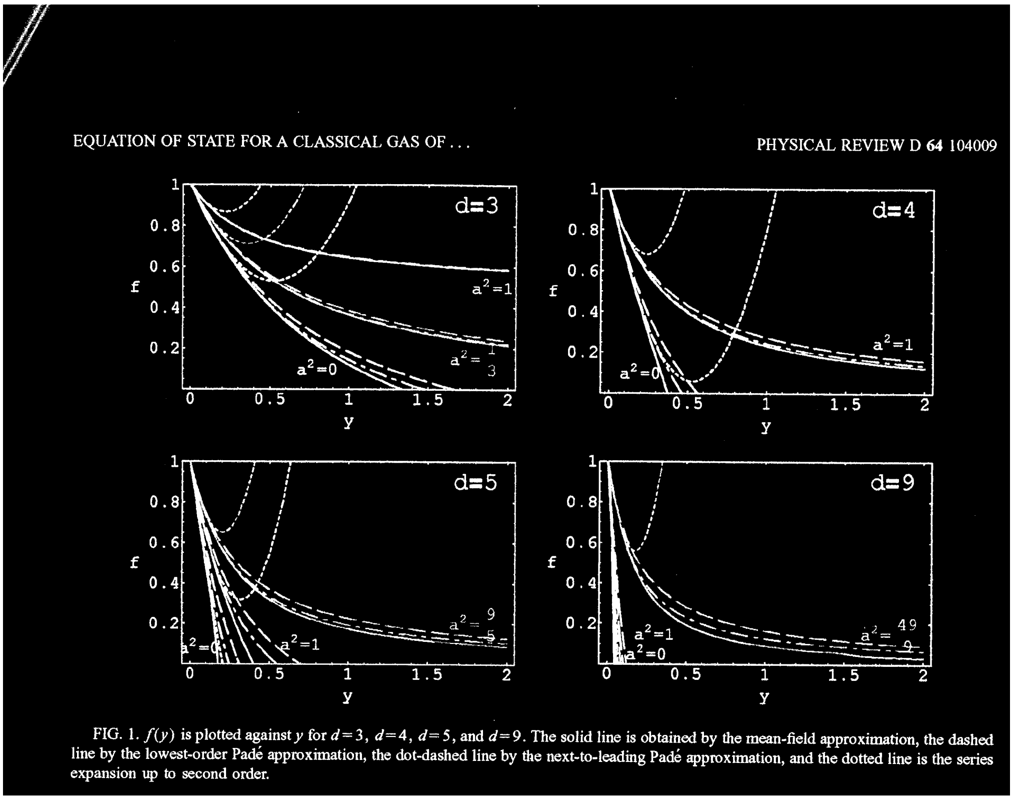



FIG. 1. $f(y)$ is plotted against $y$ for $d=3$, $d=4$, $d=5$, and $d=9$. The solid line is obtained by the mean-field approximation, the dashed line by the lowest-order Padé approximation, the dot-dashed line by the next-to-leading Padé approximation, and the dotted line is the series expansion up to second order.

************************

To obtain the moduli space metic, we examine

## Effective field theory of MCDBHs as a scalar field

Y. Degura and KS, Class. Q. Grav. 17 (2000) 4031

We consider a coupled system (extended source!)

$$S=\int d^4x\frac{\sqrt{-g}}{16\pi}\left[R-2(\nabla\phi)^2-e^{-2a\phi}F^2\right]$$
$$+\int d^4x\sqrt{-g}\left[-\varphi^* e^{-a\phi}g^{\mu\nu}(P_\mu+qA_\mu)(P_\nu+qA_\nu)\varphi-m^2e^{a\phi}\varphi^*\varphi\right]$$

$$\mathbf{q/m=(l+a^2)^{l/2}}$$

and take a Low-Energy limit ($-P_0-m=E-m<<m$, etc.)

ansatze

$$ds^2=-U^{-2}(\mathbf{x})(dt+B_i dx^i)^2+U^2(\mathbf{x})d\mathbf{x}^2$$

$$U(\mathbf{x})=[V(\mathbf{x})]^{1/(1+a^2)}$$

$$e^{-2a\phi}=[V(\mathbf{x})]^{2a^2/(1+a^2)}$$

$$A=\frac{1}{\sqrt{1+a^2}}\left\{1-[V(\mathbf{x})]^{-1}\right\}dt+A_i dx^i$$

We consider field equations up to linear order in $A_i(\mathbf{x})$ and $B_i(\mathbf{x})$.

The Dilaton field eq., (00)-component of the Einstein eq., and (0) component of the "Maxwell" eq.

$$\partial^2 V + 8\pi(1+a^2)m^2U^3|\varphi|^2 = 0$$

(0i)-component of the Einstein eq., and (i) component of the "Maxwell" eq.

$$-\frac{1+a^2}{3-a^2}\partial_j\left[V^{2(a^2-1)/(1+a^2)}\hat{F}_{ji}\right] = 8\pi q e^{-a\phi}\varphi^*(P_i + q\hat{A}_i)\varphi$$

where $$\hat{A}_i \equiv A_i + \frac{1}{\sqrt{1+a^2}}\frac{1}{V}B_i$$

We obtain the effective lagrangian density for the "non-relativistic field" $\psi \equiv \sqrt{2m}U^{3/2}\varphi$:

$$\mathcal{L} = \psi^*(-P_0 - m)\psi - \frac{1}{2mV^{(3-a^2)/(1+a^2)}}\psi^*(\mathbf{P} + q\hat{\mathbf{A}})^2\psi$$

$$+\frac{1}{16\pi}\frac{1+a^2}{3-a^2}\frac{1}{V^{2(1-a^2)/(1+a^2)}}\hat{F}^2 ,$$

which is directly applicable to the analysis of the many-body system

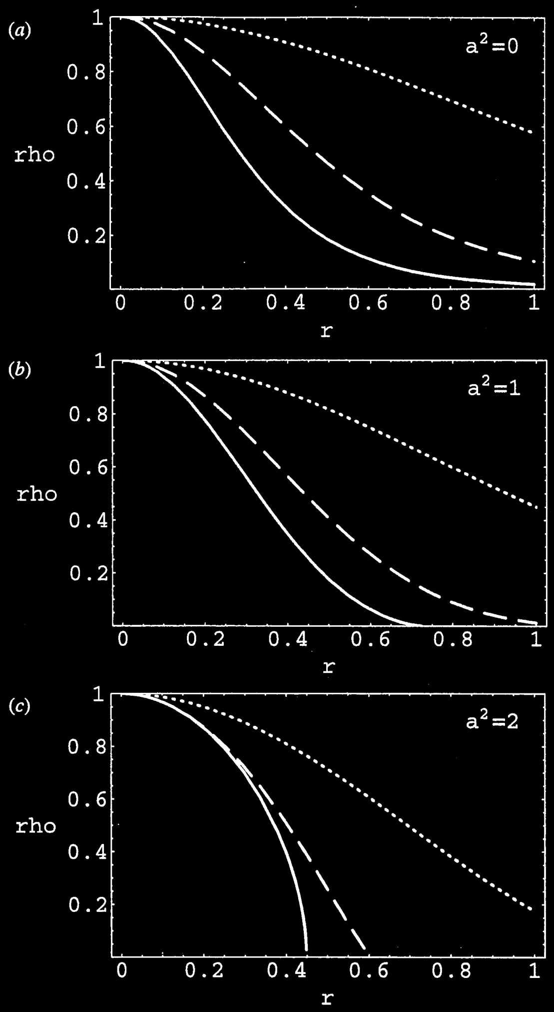



**Figure 1.** The density distribution of the isothermal sphere of 'extreme black holes' for different values of the coupling constant of the dilaton field: (*a*) $a^2 = 0$, (*b*) $a^2 = 1$ and (*c*) $a^2 = 2$. The full curve denotes $\delta = 0$ (the high-temperature limit), the broken curve $\delta = 1$ and the dotted curve $\delta = 10$.

# From the effective field theory to the moduli space geometry

$$\mathrm{H} = \frac{1}{2mV^{(3-a^2)/(1+a^2)}}\psi^*(\mathbf{P}+q\hat{\mathbf{A}})^2\psi - \frac{1}{16\pi}\frac{1+a^2}{3-a^2}\frac{1}{V^{2(1-a^2)/(1+a^2)}}\hat{F}^2$$

<u>use</u>

$$\mathbf{J} = \frac{\partial \mathrm{H}}{\partial \mathbf{P}} = \frac{1}{mV^{(3-a^2)/(1+a^2)}}\psi^*(\mathbf{P}+q\hat{\mathbf{A}})\psi = \psi^*\mathbf{v}\psi$$

$$-\frac{1+a^2}{3-a^2}\partial_1\left[V^{2(a^2-1)/(1+a^2)}\hat{F}_{1i}\right] = 4\pi q\mathbf{J}$$

$$\nabla\cdot\mathbf{J} = 0$$

$$\rho = \psi^*\psi$$

$$\partial^2 V + 4\pi(1+a^2)m\rho = 0$$

<u>and take the particle-source limit</u>

$$\mathbf{J}(\mathbf{x}) \to \sum_a \frac{m_a}{m}\mathbf{v}_a\delta^3(\mathbf{x}-\mathbf{x}_a)$$

$$\rho(\mathbf{x}) \to \sum_a \frac{m_a}{m}\delta^3(\mathbf{x}-\mathbf{x}_a)$$ etc.

Finally, we get the Hamiltonian (here, written with **v**) of O(v$^2$):

$$H = \int \mathcal{H} \; d^3\mathbf{x}$$

$$= \sum_a \frac{1}{2} m \mathbf{v}_a^2 + \frac{3-a^2}{8\pi} \int d^3\mathbf{x} [V(\mathbf{x})]^{\frac{2(1-a^2)}{1+a^2}} \sum_{ab} \frac{m_a m_b}{|\mathbf{r}_a|^3 |\mathbf{r}_b|^3}$$

$$\times \left\{ \frac{1}{2} \mathbf{r}_a \cdot \mathbf{r}_b |\mathbf{v}_a - \mathbf{v}_b|^2 - (\mathbf{r}_a \times \mathbf{r}_b) \cdot (\mathbf{v}_a \times \mathbf{v}_b) \right\}$$

with $$V(\mathbf{x}) = 1 + (1 + a^2) \sum_c \frac{m_c}{|\mathbf{r}_c|}$$

where $\mathbf{r}_a = \mathbf{x} - \mathbf{x}_a$ .

In general, above integration needs "regularization".

e.g. $\int d^3\mathbf{x} \delta^3(\mathbf{x}) / |\,\mathbf{x}\,|^p \to 0$ (p>0)

Infeld and Plebanski, Motion and Relativity (Oxford: Pergamon, 1960)

# Another form ((N+1) dim.)

$$H = \sum_{ab} \mathbf{v}^{ak} \mathbf{v}^{bl} \left( \delta_k^i \delta_l^j + \delta_{kl} \delta^{ij} - \delta_k^j \delta_l^i \right) \partial_{ai} \partial_{bj} L$$

with
$$L = -\frac{1}{32\pi} \int d^N \mathbf{x} [V(\mathbf{x})]^{\frac{2(N-1)}{N-2+a^2}}$$

where
$$V(\mathbf{x}) = 1 + \frac{2(N-2+a^2)}{(N-1)(N-2)} \frac{4\pi}{A_{N-1}} \sum_c \frac{m_c}{|\mathbf{r}_c|^{N-2}}$$

## Other General References about the BH moduli:


Gibbons and Kallosh, Phys. Rev. D51 (1995) 2839

Brooks, Kallosh and Ortin, Phys. Rev. D52 (1995) 5797

Gibbons, Papadopoulos and Stelle, Nucl. Phys. B508 (1997) 623

Kaplan and Michelson, Phys. Lett. B410 (1997) 125

Michelson, Phys. Rev. D57 (1998) 1092

Maldacena, Michelson, and Strominger, JHEP 9902 (1999) 011

Michelson and Strominger, JHEP 9909 (1999) 005

Maloney, Spradlin and Strominger, JHEP 0204 (2002) 003

Gutowski and Papadopoulos, Phys. Lett. B472 (2000) 45

Gutowski and Papadopoulos, Phys. Rev. D62 (2000) 064023

Britto-Pacumio, Michelson, Strominger and Volovich, hep-th/9911066

**In general, there are many-body (velocity-dependent) interactions among BHs.**

- **For $a^2=0$ (usual extreme RN BHs) there are 2-body, 3-body, and 4-body forces.**
- **For $a^2=1$ (string theory) there are only 2-body interactions**

  **(in any dimensions!).**
- **For $a^2=3$ (Kaluza-Klein) there is no interactions up to this order.**
- **Other Special Cases: Only 2-body and 3-body interactions exist for $a^2=1/3$.**

# 3. Two-Body Problem

*The parameter of the center of mass can be removed as the inertial motion, as usual.*

**The metric of the moduli space is:**

$$ds_{MS}^2 = \gamma(r)\left[dr^2 + r^2 d\Omega^2\right]$$

r: the distance between two BHs

where

$$\gamma(r) = \left[1 - \frac{M}{\mu} - \frac{(3-a^2)M}{r} + \frac{M}{m_1}\left(1 + \frac{(1+a^2)m_2}{r}\right)^{(3-a^2)/(1+a^2)} + \frac{M}{m_2}\left(1 + \frac{(1+a^2)m_2}{r}\right)^{(3-a^2)/(1+a^2)}\right]$$

where $M = m_1 + m_2$ : Total Mass

$\mu = m_1 m_2 / M$ : Reduced Mass

>>>>>In the limit of μ/M    0,

$$\gamma(r) \approx \left(1 + \frac{(1+a^2)M}{r}\right)^{(3-a^2)/(1+a^2)} \qquad \text{( for r>>μ)}$$

- for finite μ:

For a²=0, we recover the result of Ferrell et al.

$$\gamma(r) = 1 + \frac{3M}{r} + \frac{3M^2}{r^2} + \frac{M^3}{r^3}\left(1 - \frac{2\mu}{M}\right)$$

For $a^2=1/3$, (regardless of the value of μ)

$$\gamma(r) = \left(1 + \frac{4M}{3r}\right)^2$$

For $a^2=l$,    (regardless of the value of μ)

$$\gamma(r) = 1 + \frac{2M}{r}$$

# The Surface of the Moduli Space

We pick up the "equator" of the moduli space.

And introduce new coordinate rather than the isotropic one

$$ds^2_{MS} = \gamma(r)\left[dr^2 + r^2 d\theta^2\right]$$

$$= h(R)dR^2 + R^2 d\theta^2$$

$$= dR^2 + dz^2(R) + R^2 d\theta^2$$

where $R=\gamma^{1/2}\ r$,

$$h(R(r)) = \left(1 + \frac{r}{2\gamma}\frac{d\gamma}{dr}\right)^{-2}$$

and

$$\left(\frac{dz}{dR}\right)^2 = h - 1$$

**For $a^2<1/3$, the minimum value for R exists. The "throat" of the surface located there.**

**For $a^2=1/3$, R approaches a constant as r   0.**

**For $a^2>1/3$, the "origins" of two coordinates coincide.**

In the vicinity of the origin, R   0, the metric looks like

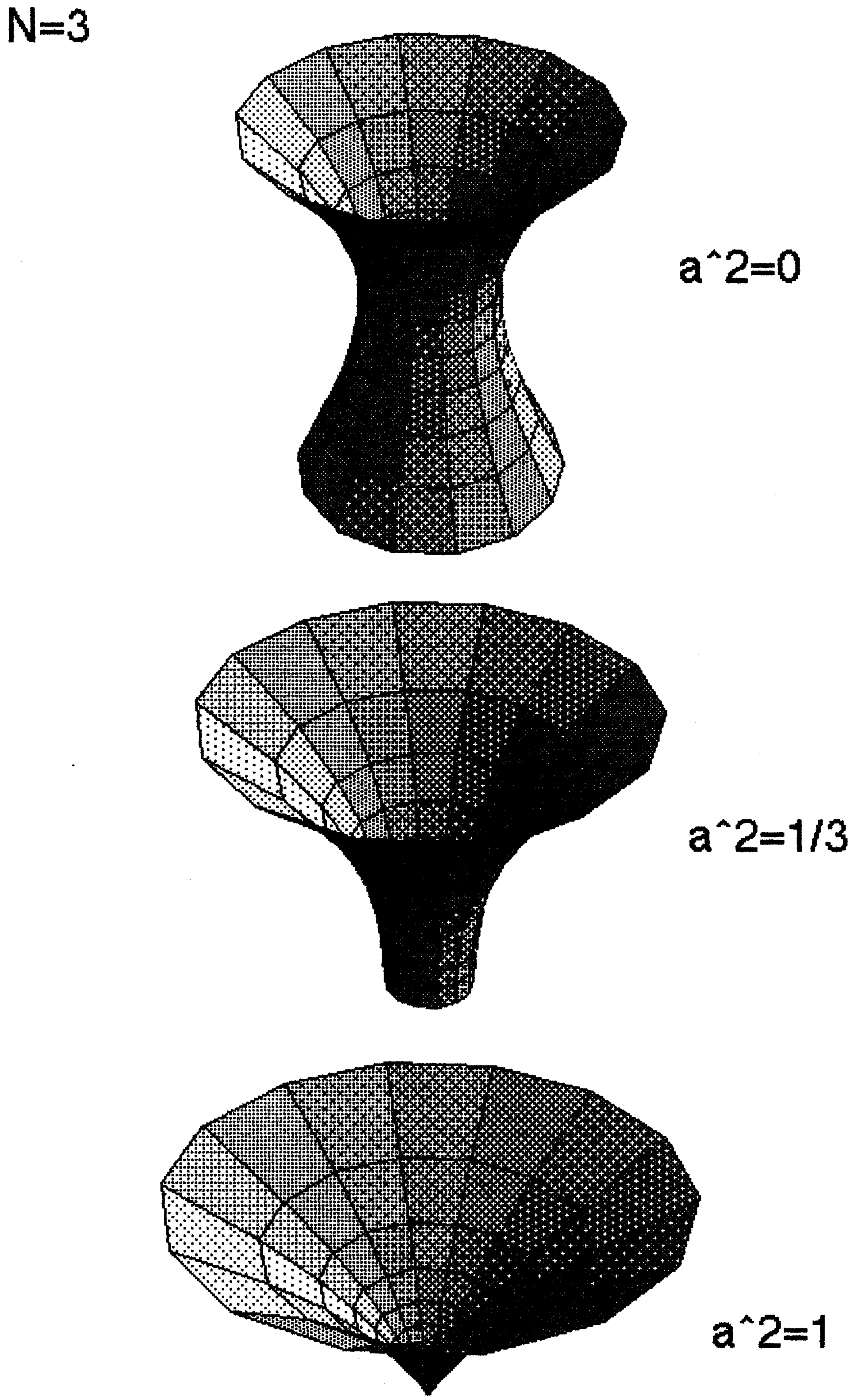
$$ds_{MS}^2 \approx \left( \frac{2(1+a^2)}{3a^2-1} \right)^2 dR^2 + R^2 d\theta^2$$

The **deficit angle** around the origin is

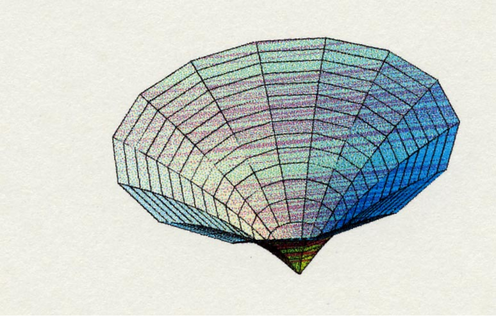
$$\Delta\theta = \frac{3-a^2}{1+a^2}\pi$$

For $a^2=1/3$, $\Delta\theta=2\pi$

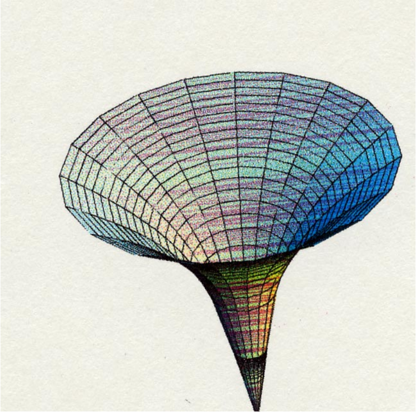
For $a^2=1$, $\Delta\theta=\pi$



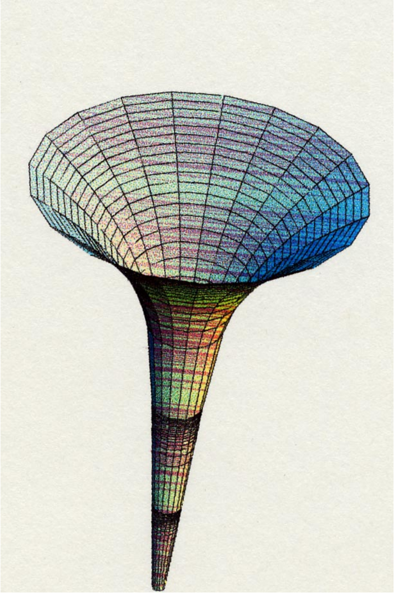
N=3
a^2=0
a^2=1/3
a^2=1



$a^2=1$



$a^2=1/2$



$a^2=2/5$

# 4. Classical Scattering

KS, Int. J. Mod. Phys. D2 (1993) 59

We can define the scattering plane for the two-body problem, so the parameters are the distance between two BHs (r) and the azimuthal angle (θ).

**For the Scattering problem, The gedesic on the moduli space is described by**

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left[\frac{\gamma(r)}{b^2} - \frac{1}{r^2}\right] \quad *$$

where

**b is <u>the impact parameter</u>.**

## Comparison with the small mass in the background geometry (Classical analysis)

**Under the assumption that one of the BH mass is much smaller than the other's, the motion of the small mass will be described by the test particle in the back ground geometry of the extreme BH.**

**Further, we can identify the coordinate of the test particle and that of the moduli space.**

The Action for the test particle is $-\int ds\left[ me^{a\phi} + qA_\mu \frac{\partial x_i^\mu}{\partial s} \right]$

where $\mathbf{q/m=(l+a^2)^{l/2}}$.

**After some manipulation, we get the equation once integrated:**

$$\left(\frac{dr}{d\theta}\right)^2 = r^4\left[\left(\frac{E-m}{L}\right)^2 V^{4/(1+a^2)} + \frac{2m(E-m)}{L^2}V^{(3-a^2)/(1+a^2)} - \frac{1}{r^2}\right]$$

where E and L are integration constants and $\mathbf{V(r)=1+(1+a^2)M/r}$

For the scattering problem, we should set $\mathbf{E=m/(1\text{-}v^2)^{1/2}}$ **and** $\mathbf{L=mvb/(1\text{-}v^2)^{1/2}}$

Then we find

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{b^2}V^{(3-a^2)/(1+a^2)} - \frac{1}{r^2} + \frac{\mathbf{v}^2}{4b^2}V^{4/(1+a^2)} + O(\mathbf{v}^4) *$$

Therefore the results of two approach coincide with each other in the small mass and low-velocity limit.

*************************

Motion of a test particle in the multicentered solution is an interesting subject to study.

In the MP background, it is known that the motion may be chaotic and the set of the initial condition may have a fractal structure if classified by the "goal" bh.

**C. P. Dettmann, N. E. Frankel and N. J. Cornish, Phys. Rev. D50 (1994) R618**

**N. J. Cornish and G. W. Gibbons, Class. Q. Grav. 14 (1997) 1865**

************************

Classical analysis of

## Coalescence of two BHs

If the right hand side of eq * never becomes zero, the trajectory ends up to the origin, i.e., two BHs coalesce.

For μ/M 0, the critical value for **b** is given as follows:

$$b_c = \frac{1-3a^2}{2}\left(\frac{3-a^2}{1-3a^2}\right)^{(3-a^2)/\{2(1+a^2)\}} M \qquad \text{for } a^2<1/3$$

$$b_c = \frac{4}{3} M \qquad \text{for } a^2=1/3$$

**for $b<b_c$, two BHs coalesce.**

・For $a^2>1/3$, the BHs never coalesce.

This can be seen from the moduli space geometry, as we have seen.

# Cross section (Classical analysis)

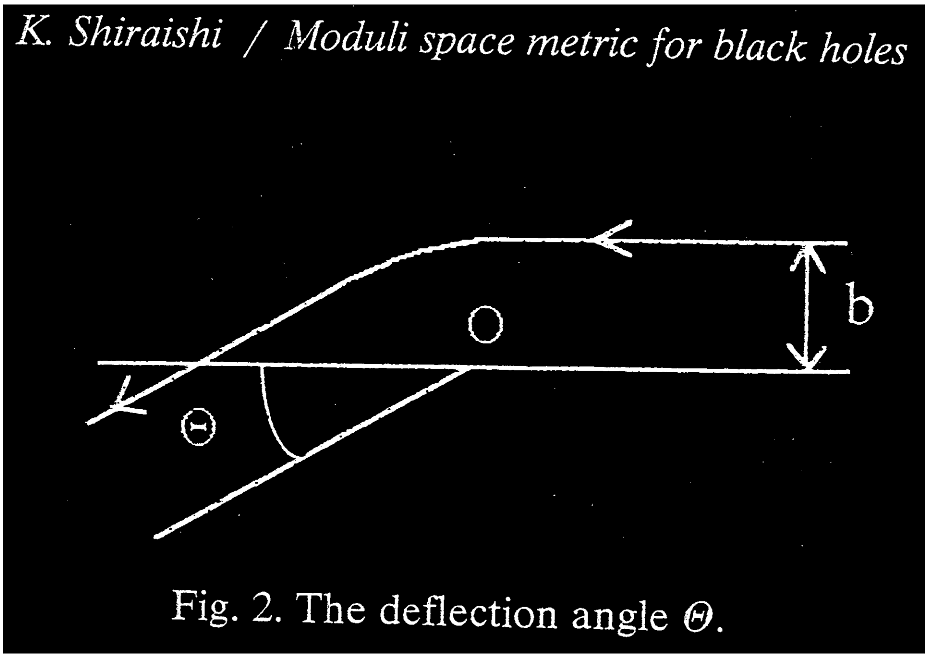



Fig. 2. The deflection angle Θ.

<u>**The deflection angle**</u> Θ is given by

$$\Theta = \int_{r_0}^{\infty} \frac{2b\,dr}{r\sqrt{r^2\gamma(r) - b^2}} - \pi$$

where $\mathbf{r_0}$ satisfies $\mathbf{r_0^2\gamma(r_0)=b^2}$.

We can express the differential cross-section using Θ.

$$\frac{d\sigma}{d\Omega} = \left| \frac{\sin\Theta}{b} \frac{d\Theta}{db} \right|^{-1}$$

For $a^2=1$,

$$\Theta = 2\tan^{-1}\frac{M}{b}$$

and

$$\frac{d\sigma}{d\Omega} = \frac{1}{4}\frac{M^2}{\sin^4(\Theta/2)}$$

This behavior (much alike **Rutherford Scattering**) can be expected by observing the geometry of the moduli space.

**--- the deficit angle=π near the origin for $a^2=1$.**

# 5. Quantum Scattering

**We consider wave function in the moduli space.**

Traschen and Ferrell, Phys. Rev. D45 (1992) 2628

The Schroedinger Equation is written as

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2\mu}\Delta+\hbar^2\xi R_{MS}\right)\Psi$$

where **Δ is the covariant Laplacian on moduli space**.

$R_{MS}$ is the curvature of the moduli space, but hereafter we consider $\xi=0$ for simplicity.

For two-body problem, the wavefunction in a stationary state for the relative motion can be factorized as

$$\Psi=\psi_{ql}(r)Y_{lm}(\theta,\varphi)\exp(-iEt/\hbar)$$

where $E=\hbar^2q^2/(2\mu)$.

The radial function is governed by

$$\psi''+\frac{2}{r}\psi'+\frac{\gamma'}{2\gamma}\psi'-\frac{l(l+1)}{r^2}\psi+q^2\gamma\psi=0 \quad **$$

************************

In close limit, the wave function can be described by Conformal Quantum Mechanics.

(K. Sakamoto and KS, Phys. Rev. D66 (2002) 024004)

************************

## Comparison with the small mass in the background geometry (Quantum mechanical analysis)

**Under the assumption that one of the BH mass is much smaller than the other's, the motion of the small mass will be described by the test wave in the back ground geometry of the extreme BH.**

**Further, we can identify the coordinate of the test wave function and that of the moduli space.**

The Wave Function for the test particle is

$$\hbar^2\left(\nabla^\mu + iqA^\mu\right)e^{-2ab\phi}\left(\nabla_\mu + iqA_\mu\right)\Psi - e^{-2ac\phi}m^2\Psi = 0$$

where **$q/m=(l+a^2)^{l/2}$** and b-c=1.

**After some manipulation, we get the equation for the radial function:**

$$\Psi'' + \frac{2}{r}\Psi' + \frac{X'}{2X}\Psi' - \frac{l(l+1)}{r^2}\Psi + V^{4/(1+a^2)}\frac{1}{\hbar^2}\left[(E-m)^2 + \frac{2m(E-m)}{V}\right]\Psi = 0$$

where $X= e^{-4ab\phi}$.

In the non-relativistic and low-energy limit, $\hbar q << m$, above equation becomes

$$\Psi'' + \frac{2}{r}\Psi' + \frac{X'}{2X}\Psi' - \frac{l(l+1)}{r^2}\Psi = -q^2V^{(3-a^2)/(1+a^2)}\Psi + O(q^4)$$

**Therefore the results of two approach coincide with each other in the small mass and low-velocity limit, up to the coefficient of the first derivative term.**

## Effective Potential

The equation ** can be written in a simple form by introducing a new coordinate

$$R = \int \sqrt{\gamma}\, dr$$

and writing

$$\Psi = \frac{\chi}{r\sqrt{\gamma}}$$

Then one finds

$$\chi_{,RR} + (q^2 - V_{eff})\chi = 0$$

where

$$V_{eff} = \frac{1}{2r\gamma}\left(\frac{r\gamma'}{\gamma}\right)' + \frac{l(l+1)}{r^2}$$

**The range of R**

For $a^2 \le 1/3$, [-∞, +∞]

For $a^2 > 1/3$, [0, +∞]

.3)
2.
he
en
ier
to
ite
he
he
to
nd
he
is
ial
m,
.4)
as
.5)
ai-
re-
w-
be
ity



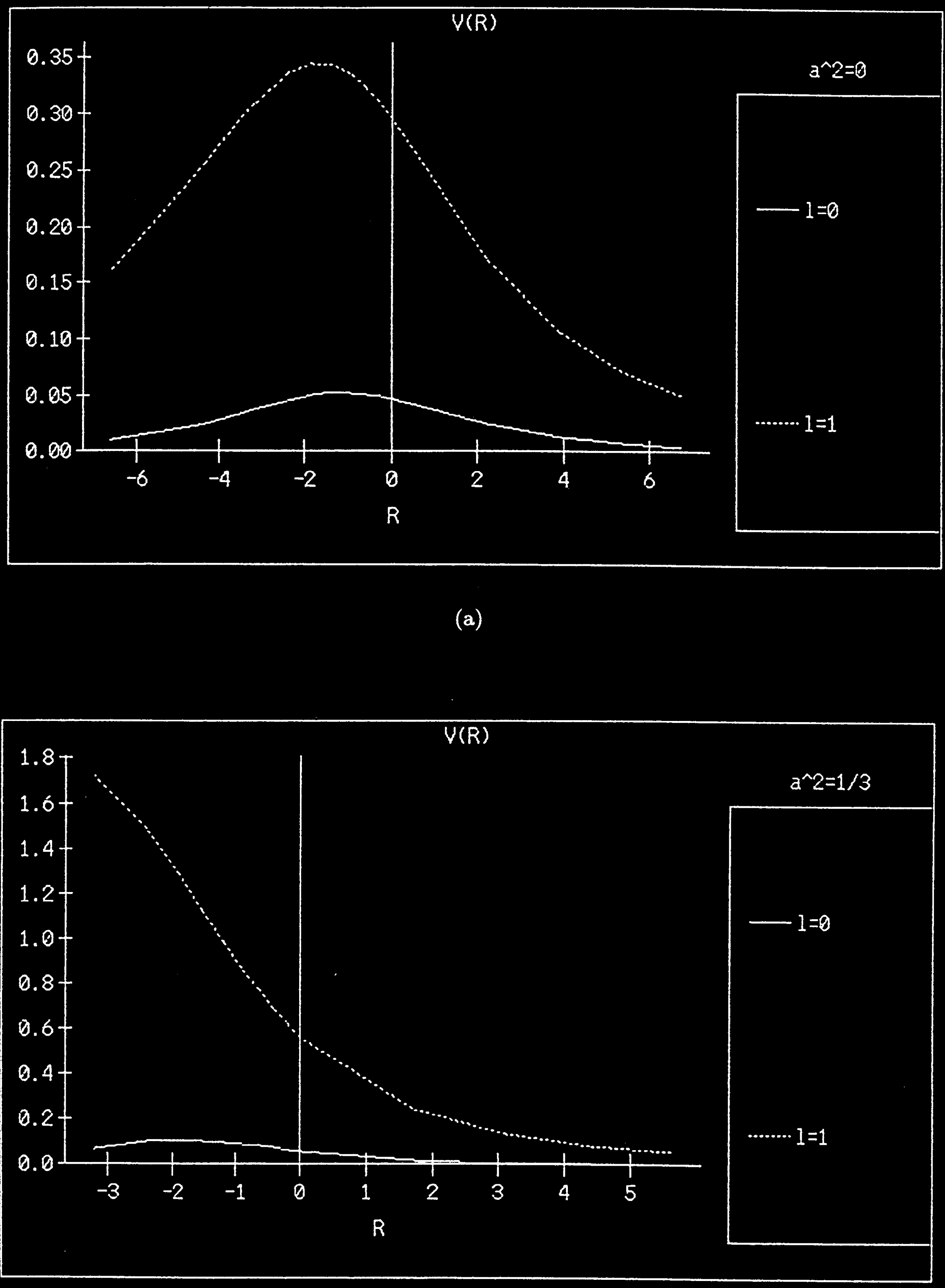
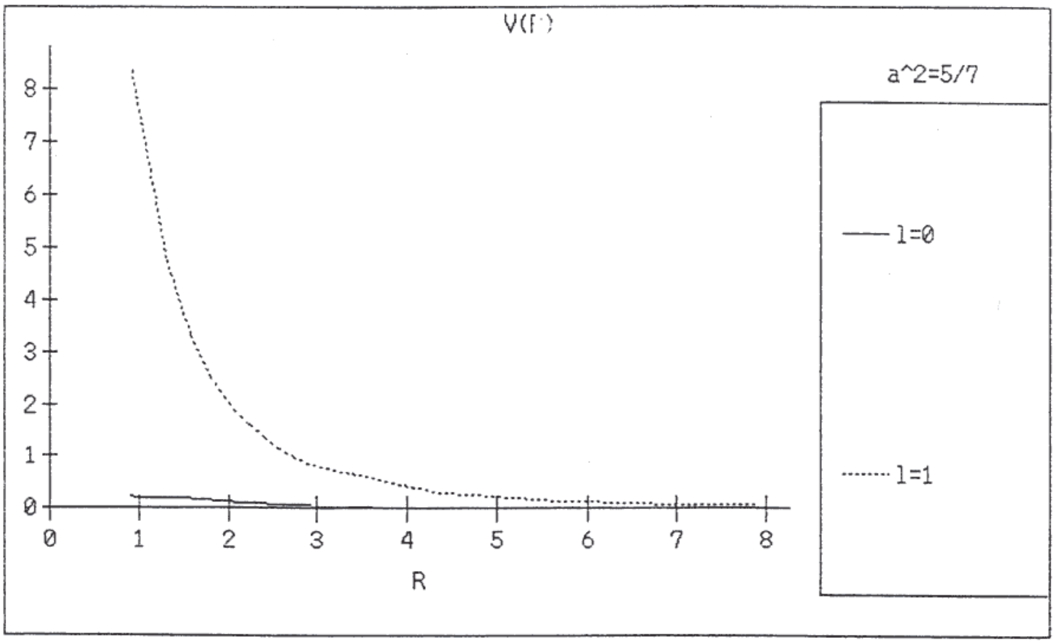
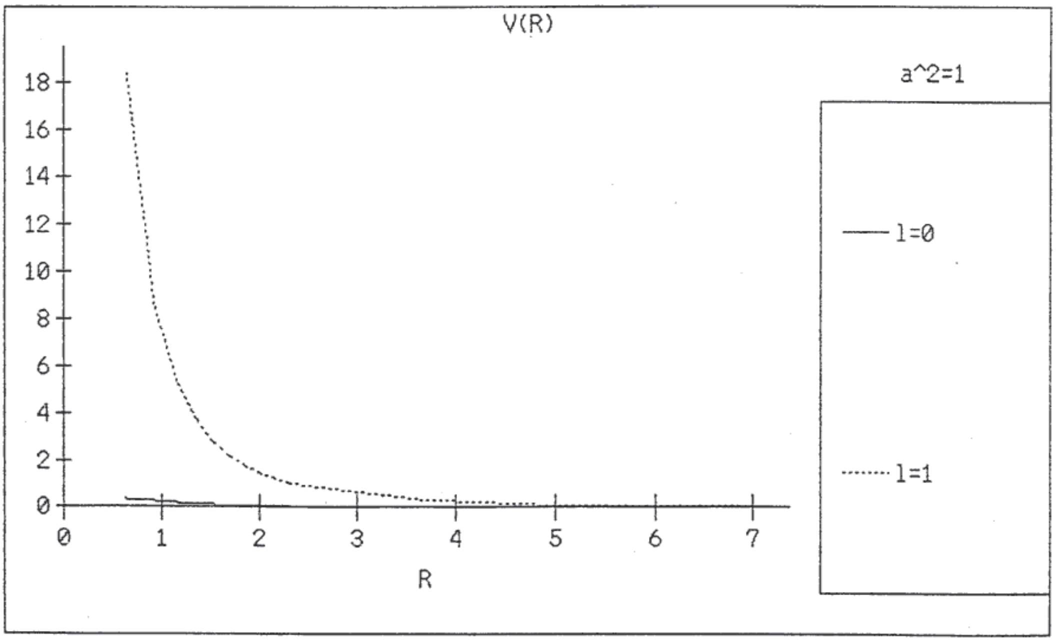



Fig. 1. The potential $V(R)$ is plotted against $R$, for several cases. (a) $a^2 = 0$, (b) $a^2 = 1/3$, (c) $a^2 = 5/7$, (d) $a^2 = 1$ and (e) $a^2 = 7/5$. The $l = 0$ and $l = 1$ cases are shown in each figure. Here, $R$ is normalized by $r_+$ and $V(R)$ is normalized by $(r_+)^{-2}$, with $r_+ = (1 + a^2)M$.

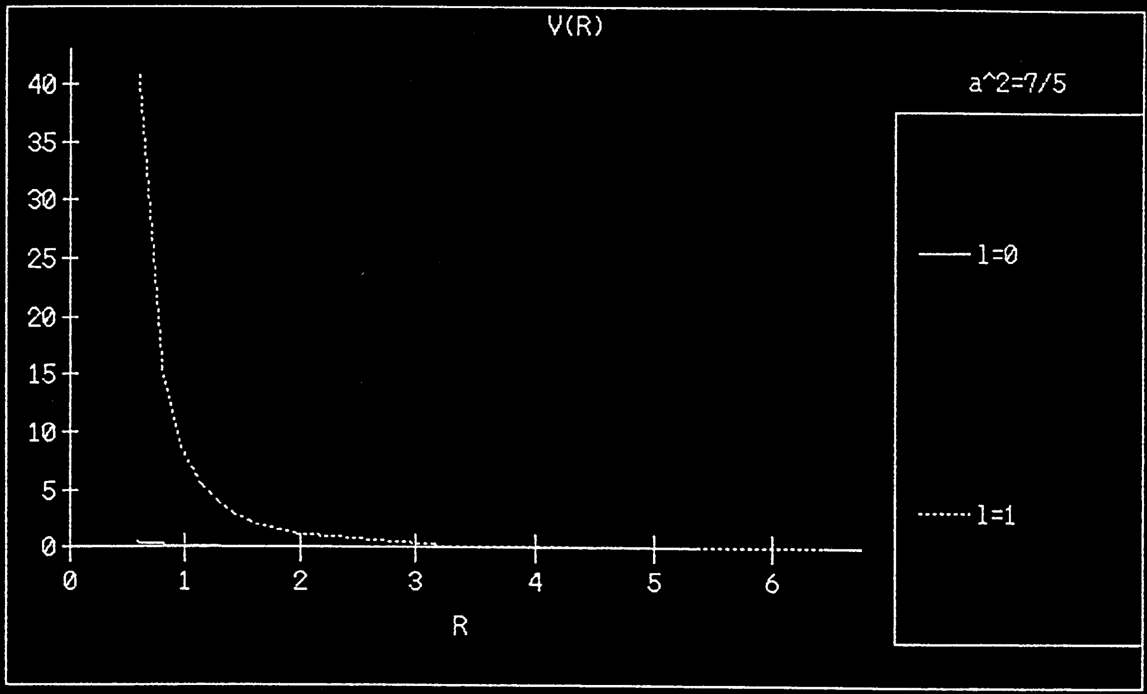



(c)




(d)

Fig. 1. (*Continued*)




(e)

Fig. 1. (*Continued*)

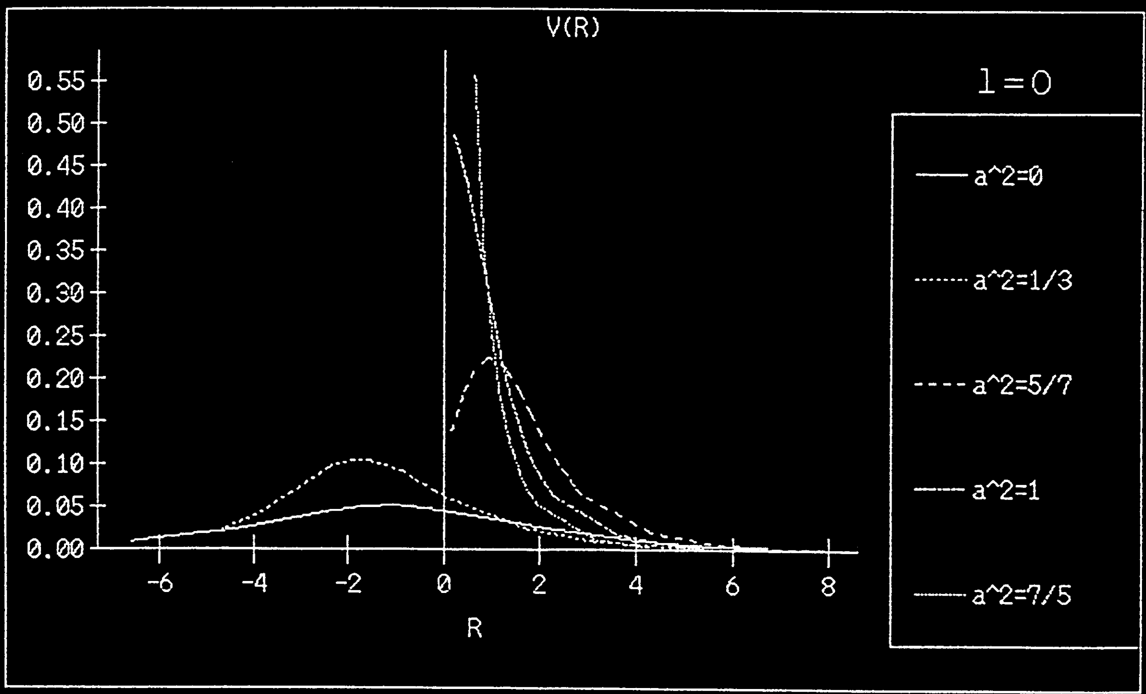



Fig. 2. The potential $V(R)$ for $l = 0$ is plotted against $R$ in a graph. The normalization is the same as in Fig. 1.

# FIGURES

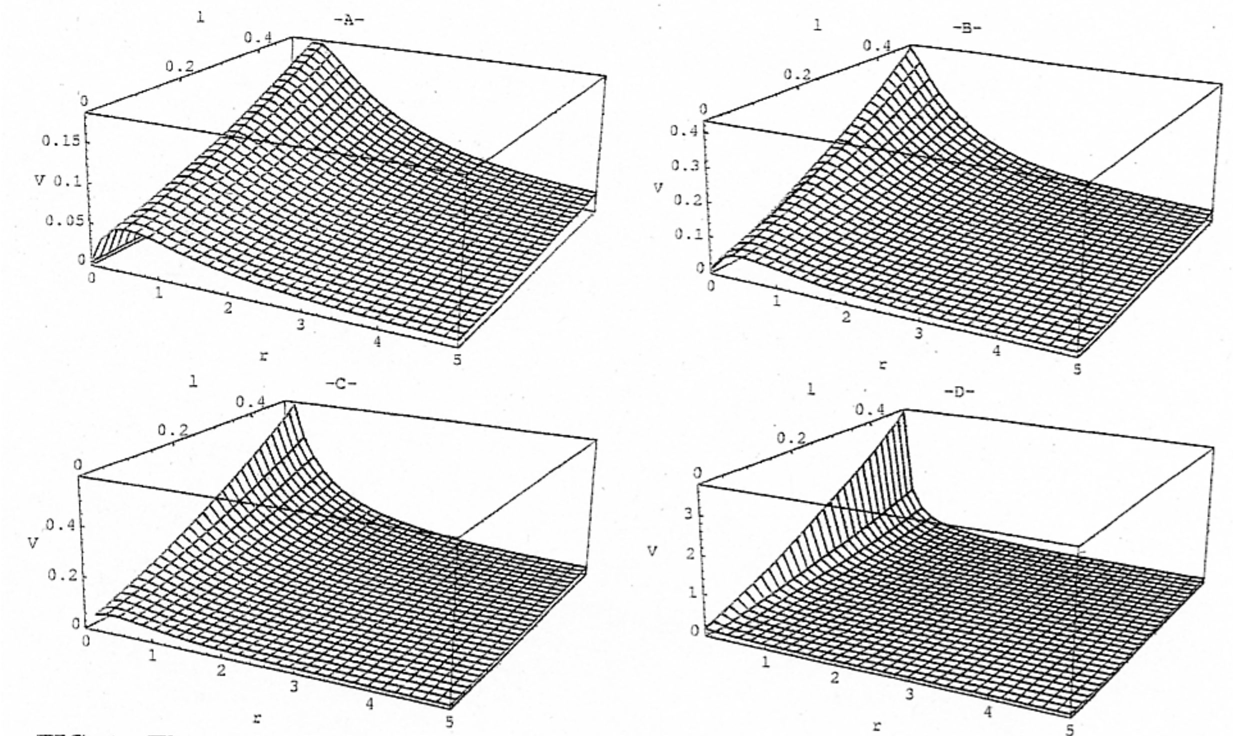



FIG. 1. The potential $V$ as the function of $r$ and angular momentum $l$, A for $a^2 = 0$, B for $a^2 = 1/3$, C for $a^2 = 1/2$ and D for $a^2 = 1$.

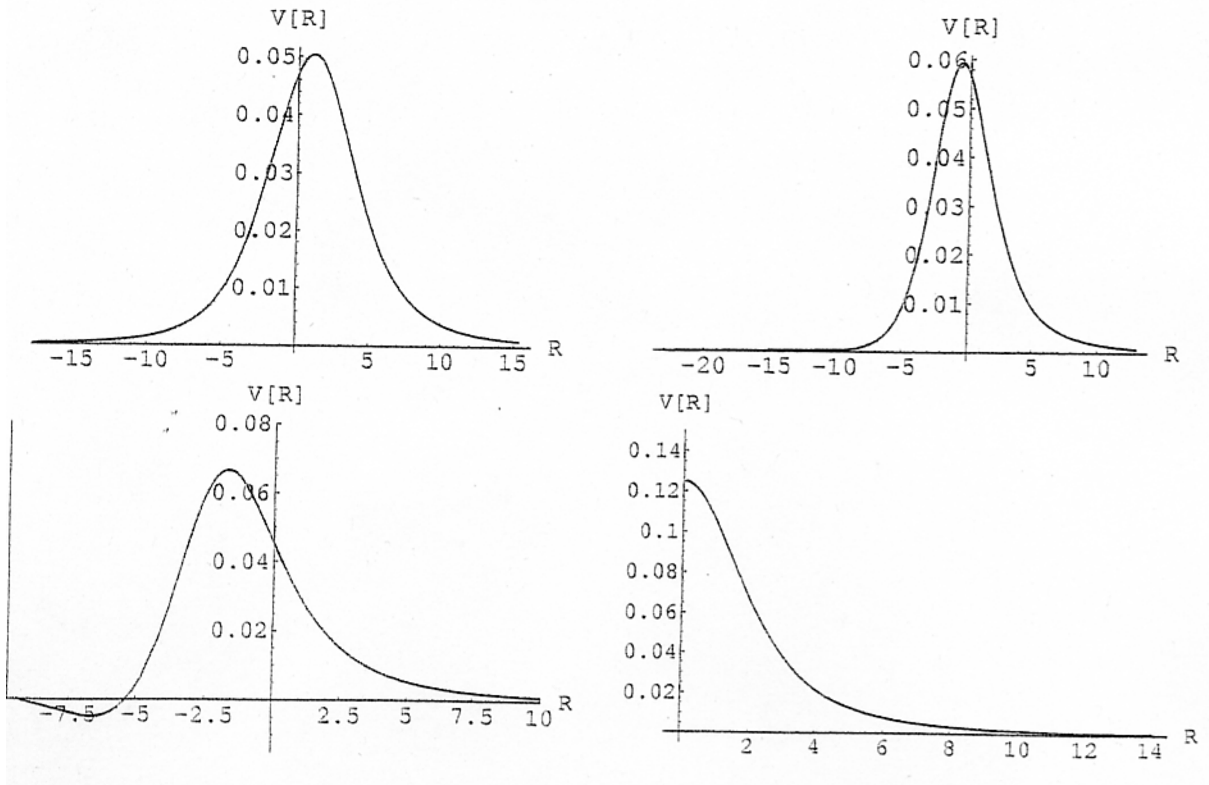



FIG. 2. The potential $V$ as the function of $R$ when the angular momentum $l = 0$, A for $a^2 = 0$, B for $a^2 = 1/3$, B for $a^2 = 1/2$ and D for $a^2 = 1$.

# WKB approximation

Phase shift

$$\delta_l = -qR_0 + \int_{R_0}^{\infty}\left(\sqrt{q^2 - V_{eff} - \frac{1}{4R^2}} - q\right) + \frac{2l+1}{4}\pi$$

$$\left(V_{eff}(R_0) = q^2 - \frac{1}{4R_0^2}\right)$$

deflection angle

$$\Theta = \pi + \int_{R_0}^{\infty} dR \frac{\partial}{\partial l}\sqrt{q^2 - V_{eff} - \frac{1}{4R^2}}$$

simple exercises!

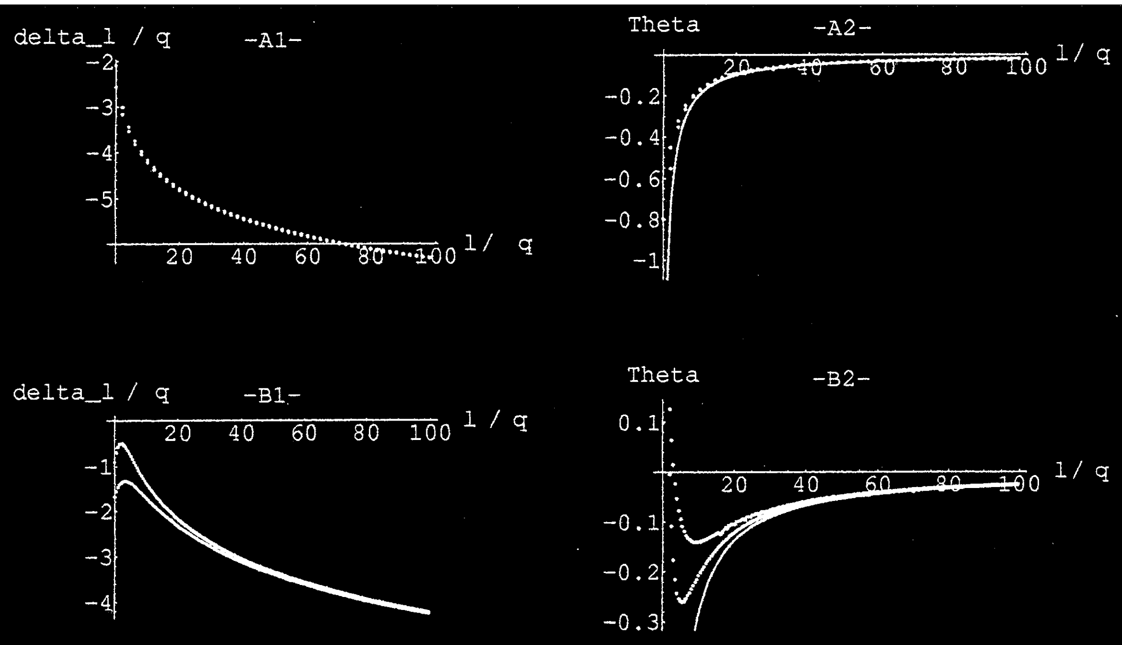



FIG. 3. The phase shift and the deflection angle. A1 is a phase shift for $a^2 = 1$, A2 is a deflection angle for $a^2 = 1$, B1 and B2 are phase shift and deflection angle for $a^2 = 1/3$.

Quantum Mechanical analysis of

**Coalescence of two BHs (for** $a^2 \leq 1/3$**)**

**The Effective Potential has a <u>Maximum</u> Value.**

**In the limit μ/M    0,**

$$V_{\max} \approx \frac{4l(l+1)}{1-3a^2}\left(\frac{1-3a^2}{3-a^2}\right)^{(3-a^2)/(1+a^2)}$$

**The wave goes over the top and is absorbed**

**(    R=-∞), i.e. two BHs coalesce. This occurs when**

$$b \approx \frac{\sqrt{l(l+1)}}{q} < \frac{1-3a^2}{2}\left(\frac{3-a^2}{1-3a^2}\right)^{(3-a^2)/\{2(1+a^2)\}} M$$

For $a^2$>l/3, the BHs never coalesce.

This can be seen from the Effective Potential.

For $1/3<a^2<l$, there seem to exist **quasi-stable states**.

(This means the long stay in the vicinity of the BH)

(K. Sakamoto and KS)

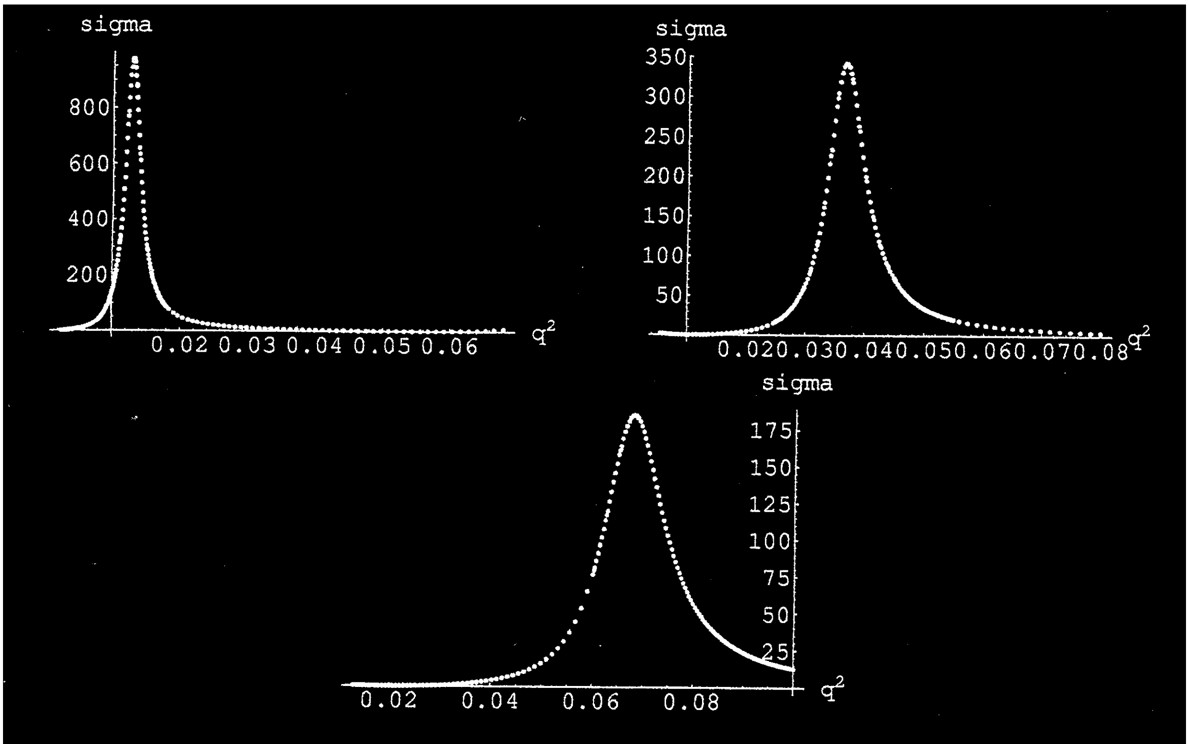



FIG. 5. The partial cross-section for $l = 0$, A for $a^2 = 2/5$, B for $a^2 = 9/20$ C for $a^2 = 1/2$.

**(we used the WKB method of Brink and Takigawa, Nucl. Phys. A279 (1977) 159)**

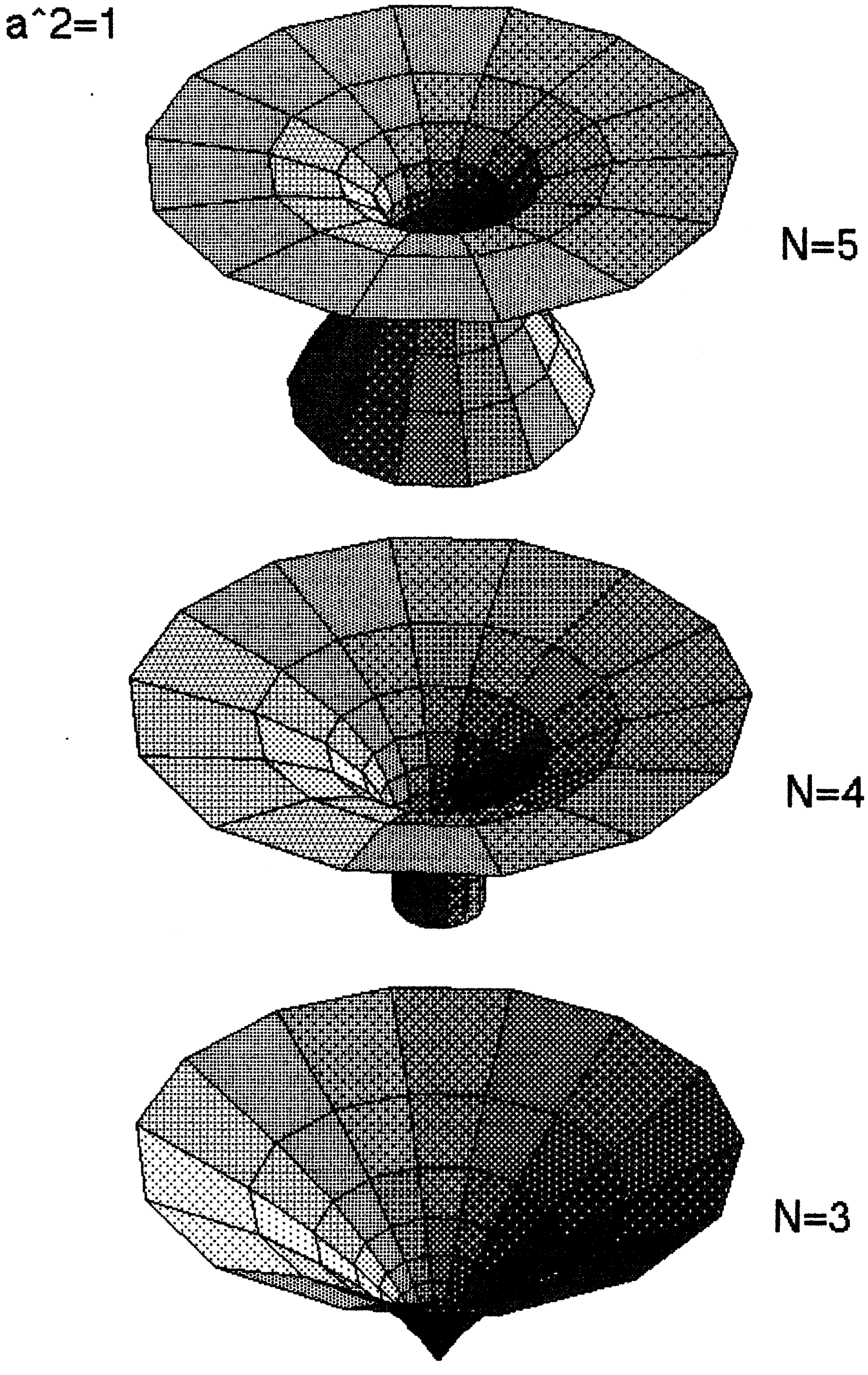
# Summary

**We have studied the interaction among the maximally-charged dilatonic BHs in the low-velocity limit.**

**The nature of the interaction depends on the value of the dilaton coupling "a", in quality as well as in quantity.**

**a=1 is a special value, (in any dimensions), for realizing a simple 2-body interactions.**
**There is another critical value, for coalescence of two MCDBHs: it is $a^2=1/3$.**

**The geodesic approximation is fairly justified in the small mass and low velocity limit, by using the test particle/wave analyses.**




# Moduli Surfaces in other dimensions